

A data-driven nonparametric specification test for dynamic regression models ¹

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This version: May 2005

Abstract

The paper introduces a new nonparametric specification test for dynamic regression models. The test combines Chi-square statistics based on Fourier series regression. A data-driven choice of the regression order, which uses the square-root of the number of Fourier coefficients, is proposed. The benefits of the new test are: (1) the selection procedure produces explicit and Chi-square critical values which give a finite sample size close the nominal size; (2) the test is adaptive rate-optimal and detects local alternatives converging to the null with a rate that can be made arbitrarily close to the parametric rate. Simulation experiments illustrate the practical relevance of the new test.

JEL Classification: Primary C52; Secondary C14.

Keywords: Specification of dynamic regression models; Data-driven nonparametric tests; Adaptive nonparametric tests; Dependent data.

¹The first author would like to acknowledge financial support from the Fonds Québécois de la Recherche sur la Société et la Culture (FQRSC). The second author would like to acknowledge financial support from LSTA.

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1 Introduction

Starting with Bierens (1984) and Robinson (1989), nonparametric specification testing for dependent data has received much attention in the econometric literature. The range of potential applications includes nonlinearity tests and time series model building as reviewed in Tjøstheim (1994) and Fan and Yao (2003), specification of continuous-time diffusion model for interest rates (Aït-Sahalia (1996)), specification of the Phillips curve (Hamilton (2001)), rational expectations models and conditional portfolio efficiency (Chen and Fan (1999), Robinson (1989)), tests of the Black and Scholes formula (Aït-Sahalia, Bickel and Stocker (2001)) among others.

An important branch of this literature has considered a nonparametric approach which uses a smoothing parameter, as a bandwidth or the order of a series expansion. This has raised two important issues, regarding the detection properties and the size accuracy. The former can be addressed with efficiency considerations, as pioneered in Ingster (1992, 1993), see also Guerre and Lavergne (2002). This framework leads to calibrate tests to detect alternatives, in a given smoothness class, that approach the null at the fastest possible rate. However, the proposed smoothing parameters depend upon the chosen smoothness class, which is too restrictive for practical applications since the choice of a smoothness class is often arbitrary. Regarding the size issue, the statistics considered in the literature are often quadratic but the critical values are computed from a normal approximation which may be inaccurate, see Hong and White (1995) for nonparametric series and Tjøstheim (1994) for kernel methods. Recent work for i.i.d. observations, such as Fan, Zhang and Zhang (2001), suggests that more sophisticated approximations should be used instead of the normal. Härdle and Mammen (1993) and Gozalo (1997) have proposed, among others, bootstrapped critical values as a solution. This may be difficult when the parametric model under consideration is specified in continuous-time and is therefore costly to simulate or to bootstrap. Bootstrapping is also a burden when the dynamic specification includes covariates which are not strongly exogenous and need to be simulated.

An important step for the detection issue was the development of the adaptive framework. Under this approach, the smoothness class containing the alternative is considered as unknown. Adaptive tests combine several statistics, designed for a specific class, to build a test, see Hart (1997) for a review of earlier work in this direction. Spokoiny (1996) has developed an efficiency theory for the adaptive case. Various papers considered adaptive rate-optimal tests using the maximum of the statistics, including Fan (1996), Fan and Huang (1999), Horowitz and Spokoiny (2001), Spokoiny (1996,2001). More specifically, Horowitz and Spokoiny (2001) have proposed an adaptive rate-optimal kernel-based specification test for a general parametric regression model that has generated various extensions. Baraud, Huet and Laurent (2003) consider some nonasymptotic refinements of the maximum approach for specification of a linear model. Poo, Sperlich and Vieu (2004) are interested in a semiparametric null hypothesis while Gayraud and Pouet (2003) considered a nonparametric null. Gao and King (2001,2004) and Fan and Yao (2003) have proposed to

extend the scope of applications to dependent data.

However, the maximum approach produces statistics with unstable asymptotic null behavior, so that achieving an accurate size remains a difficult issue. Fan (1996) found that the null limit distribution of his test gives a poor approximation for finite samples. Horowitz and Spokoiny (2001) did not derive a null limit distribution and used simulated critical values. On the other hand, Guerre and Lavergne (2005) built on a data-driven selection procedure which, under the null, selects a prescribed statistic with a high probability. Compared to the maximum approach, this considerably reduces the complexity of the null behavior of the resulting test statistic, which asymptotic distribution is a standard normal given by a specific statistic. But the statistics of Guerre and Lavergne (2005) have a complicated quadratic structure so that these authors used bootstrapped critical to achieve a level close to the nominal size. Hence, as earlier mentioned, such an approach may not be suitable for dynamic model.

In this paper, a suitable modification of the Guerre and Lavergne (2005) test is proposed to derive an adaptive rate-optimal specification test with an accurate size in a dynamic setting. The considered null hypothesis is the specification of the conditional mean for a time series with heteroskedastic innovations. Nonparametric series methods are used to compute Chi-square statistics of various orders, which, in case of low degrees of freedom, have an accurate Chi-square approximation under the null. A selection criterion, using a low penalty term proportional to the square root of the number of coefficients, chooses a test statistic. Hence the rejection region of the test can use accurate Chi-square critical values. The rest of the paper is organized as follows. Section 2 presents our test and the adaptive framework on a non technical level. Section 3 groups our main assumptions and our main results. After studying the null behavior of the test, adaptive rate-optimality is introduced and the test is shown to be efficient. Detection of local alternatives, approaching the null with a rate close to the parametric one, is also considered. Section 4 illustrates the size and detection properties of the test with a simulation experiment and Section 5 concludes the paper. The proofs are grouped in Section 6 and two appendices.

2 Heuristics of the data-driven test

Consider an autoregressive model with exogenous variables Z_t ,

$$Y_t = \mu(Y_{t-1}, \dots, Y_{t-q}, Z_t) + \varepsilon_t = \mu(X_t) + \varepsilon_t$$

with $X_t = [Y_{t-1}, \dots, Y_{t-q}, Z_t']' \in \mathbb{R}^d$, $\mathbb{E}[\varepsilon_t | \mathcal{F}_t] = 0$ and $\text{Var}[\varepsilon_t | \mathcal{F}_t] = \sigma^2(X_t)$, where \mathcal{F}_t is the past Borel field generated by X_1, \dots, X_t . Given T observations $(Y_1, X_1), \dots, (Y_T, X_T)$, we want to test that $\mu(\cdot)$ belongs to some parametric family $\{m(\cdot; \theta), \theta \in \Theta \in \mathbb{R}^p\}$, that is the correct specification hypothesis

$$H_0: \mu(\cdot) = m(\cdot; \theta) \quad \text{for some } \theta \in \Theta.$$

The proposed procedure builds on the estimated residuals $\widehat{U}_t = Y_t - m(X_t; \widehat{\theta}_T)$, where $\widehat{\theta}_T$ is a consistent estimator of θ under H_0 , as for instance the nonlinear least squares estimator

$$\widehat{\theta}_T = \arg \min_{\theta \in \Theta} \sum_{t=1}^T (Y_t - m(X_t; \theta))^2.$$

By $Y_t = \mu(X_t) + \varepsilon_t$, the residuals decompose as $\widehat{U}_t = \widehat{\Delta}(X_t) + \varepsilon_t$, where $\widehat{\Delta}(\cdot) = \mu(\cdot) - m(\cdot; \widehat{\theta}_T)$ indicates potential misspecification, which asymptotically vanishes under the null but not under the alternative. Our test combines nonparametric series statistics constructed by projecting the residuals to detect the presence of a significant $\widehat{\Delta}(\cdot)$ over a compact $\Lambda = [-\lambda, \lambda]^d$. More specifically, we focus on multivariate Fourier series regression.¹ For $k = [k_1, \dots, k_d]' \in \mathbb{Z}^d$, define the k -th trigonometric function over Λ as

$$\psi_k(x) = \prod_{\ell=1}^d F_{k_\ell}(x_\ell) \text{ where } F_n(z) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\lambda}} \cos\left(\frac{\pi n z}{\lambda}\right) \mathbb{I}(-\lambda \leq z \leq \lambda) & n < 0, \\ \frac{1}{\sqrt{2\lambda}} \mathbb{I}(-\lambda \leq z \leq \lambda) & n = 0, \\ \frac{\sqrt{2}}{\sqrt{\lambda}} \sin\left(\frac{\pi n z}{\lambda}\right) \mathbb{I}(-\lambda \leq z \leq \lambda) & n > 0, \end{cases} \quad (2.1)$$

so that $\{\psi_k(\cdot), k \in \mathbb{Z}^d\}$ is an $L_2(dx)$ -orthonormal system, that is $\int_{\Lambda} \psi_k(x) \psi_{k'}(x) dx = 1$ if $k = k'$ and 0 otherwise. Let $|k| = \sum_{\ell=1}^d |k_\ell|$ be the degree of $\psi_k(\cdot)$. The series estimation of $\widehat{\Delta}(\cdot)$ over Λ builds on trigonometric multivariate polynomial function $\sum_{|k| \leq K} b_k \psi_k(\cdot)$ of degree K , with a number c_K of coefficients b_k proportional to K^d . To account for heteroskedasticity, assume that an estimator $\widehat{\sigma}(\cdot)$ of $\sigma(\cdot)$ is given, and consider the generalized least squares estimator $\widehat{\beta}_K = [\widehat{b}_k, |k| \leq K]'$,

$$\widehat{\beta}_K = \left(\Psi'_K \widehat{\Omega}^{-1} \Psi_K \right)^{-1} \Psi'_K \widehat{\Omega}^{-1} \widehat{U} = \arg \min_{[b_k, |k| \leq K]'} \sum_{t=1}^T \left(\frac{\widehat{U}_t - \sum_{|k| \leq K} b_k \psi_k(X_t)}{\widehat{\sigma}(X_t)} \right)^2,$$

where $\widehat{U} = [\widehat{U}_1, \dots, \widehat{U}_T]'$, $\widehat{\Omega}^{1/2}$ is the diagonal matrix with entries $\widehat{\sigma}(X_t)$, and Ψ_K is the $T \times c_K$ matrix $[\psi_k(X_t), 1 \leq t \leq T, |k| \leq K]$. Suppose that $\widehat{\Delta}(\cdot)$ is a trigonometric polynomial function of order K . A standard procedure to test the significance of Fourier coefficients would use the Chi-square statistic

$$\widehat{R}_K = \widehat{U}' \widehat{\Omega}^{-1} \Psi_K \left(\Psi'_K \widehat{\Omega}^{-1} \Psi_K \right)^{-1} \Psi'_K \widehat{\Omega}^{-1} \widehat{U} = \sum_{t=1}^T \left(\frac{\sum_{|k| \leq K} \widehat{b}_k \psi_k(X_t)}{\widehat{\sigma}(X_t)} \right)^2, \quad (2.2)$$

leading to reject H_0 when \widehat{R}_K is large. However, assuming that $\widehat{\Delta}(\cdot)$ has a finite series expansion of known order K is too simplistic for practical applications. More generally, an arbitrary choice of K may affect the power and a better understanding of the impact of K is important to build a proper specification test. Set

¹Using other series approximation methods, as for instance polynomial functions or wavelets, is possible but leads to a more involved theoretical study. Indeed, the Fourier system satisfies $\sup_{k \in \mathbb{Z}^d} \sup_{x \in \Lambda} |\psi_k(x)| < \infty$, a condition which simplifies algebraic manipulations under dependence mixing conditions. Another interest of Fourier methods is that using wavelets may limit the scope of applications to alternatives with a maximal smoothness given by the choice of the wavelet basis, see the wavelet tests considered in Spokoiny (1996) and Theorem 2.4 therein.

$\widehat{\Delta} = [\widehat{\Delta}(X_1), \dots, \widehat{\Delta}(X_T)]'$ and $\varepsilon = [\varepsilon_1, \dots, \varepsilon_T]'$ so that $\widehat{U} = \widehat{\Delta} + \varepsilon$ and \widehat{R}_K decomposes into three terms $\widehat{R}_K = \widehat{R}_{1K} + 2\widehat{R}_{2K} + \widehat{R}_{3K}$ with:

$$\begin{aligned}\widehat{R}_{1K} &= \widehat{\Delta}'\widehat{\Omega}^{-1}\Psi_K \left(\Psi_K'\widehat{\Omega}^{-1}\Psi_K \right)^{-1} \Psi_K'\widehat{\Omega}^{-1}\widehat{\Delta}, \\ \widehat{R}_{2K} &= \widehat{\Delta}'\widehat{\Omega}^{-1}\Psi_K \left(\Psi_K'\widehat{\Omega}^{-1}\Psi_K \right)^{-1} \Psi_K'\widehat{\Omega}^{-1}\varepsilon, \\ \widehat{R}_{3K} &= \varepsilon'\widehat{\Omega}^{-1}\Psi_K \left(\Psi_K'\widehat{\Omega}^{-1}\Psi_K \right)^{-1} \Psi_K'\widehat{\Omega}^{-1}\varepsilon.\end{aligned}\tag{2.3}$$

The term \widehat{R}_{1K} is crucial regarding detection of potential misspecification. It is the squared norm of the orthogonal projection of $\widehat{\Omega}^{-1/2}\widehat{\Delta}$ on the columns of $\widehat{\Omega}^{-1/2}\Psi_K$, which increases with K up to $\sum_{t=1}^T \widehat{\Delta}^2(X_t)\mathbb{I}(X_t \in \Lambda)/\widehat{\sigma}^2(X_t)$, achieved for $c_K \geq T$. Hence \widehat{R}_{1K} can be viewed as an downward biased estimation of the empirical measure of misspecification $\sum_{t=1}^T \widehat{\Delta}^2(X_t)\mathbb{I}(X_t \in \Lambda)/\widehat{\sigma}^2(X_t)$, that is

$$\widehat{R}_{1K} = \sum_{t=1}^T \frac{\widehat{\Delta}^2(X_t)}{\widehat{\sigma}^2(X_t)} \mathbb{I}(X_t \in \Lambda) + \text{bias}_\mu(K),$$

where $\text{bias}_\mu(K) \leq 0$ depends upon the unknown $\mu(\cdot)$ and decreases with K . The other important term in the decomposition (2.3) of the statistic \widehat{R}_K is \widehat{R}_{3K} , a pure noise term. It can be expected that \widehat{R}_{3K} is asymptotically a Chi-square variable with c_K degree of freedom, with mean c_K and variance $2c_K$, so that $\widehat{R}_{3K} = c_K + \sqrt{2c_K}O_{\mathbb{P}}(1)$. Neglecting² \widehat{R}_{2K} and substituting in (2.3) gives a bias variance type decomposition for $\widehat{R}_K - c_K$ ³

$$\widehat{R}_K - c_K = \sum_{t=1}^T \frac{\widehat{\Delta}^2(X_t)}{\widehat{\sigma}^2(X_t)} \mathbb{I}(X_t \in \Lambda) + \text{bias}_\mu(K) + \sqrt{2c_K}O_{\mathbb{P}}(1).\tag{2.4}$$

Looking for the best estimator $\widehat{R}_K - c_K$ of the misspecification indicator suggests that an ideal choice of K should achieve the minimum of $|\text{bias}_\mu(K)| + \sqrt{2c_K}O_{\mathbb{P}}(1)$. However, this is infeasible in practice, at least because $\text{bias}_\mu(\cdot)$ depends upon the unknown $\mu(\cdot)$. Alternative feasible choices of K include AIC and BIC as reviewed in Hart (1997). These selection procedures consider a K achieving the maximum of $\widehat{R}_K - \gamma c_K$ where γ is a penalty parameter. According to (2.4), this amounts to achieve the minimum of $|\text{bias}_\mu(K)| + (\gamma - 1)c_K(1 + o_{\mathbb{P}}(1))$. Therefore these selection procedures asymptotically balance $|\text{bias}_\mu(K)|$ with $(\gamma - 1)c_K$ in place of the ideal order $c_K^{1/2}$ in (2.4). This suggests to use instead a lower penalty term of the form $c_k + \gamma c_k^{1/2}$ affecting the square root of the number of coefficients $c_K^{1/2}$ in place of c_k . More specifically,

²Assume that H_0 is $\mu(\cdot) = 0$ and that $\sigma(\cdot)$ is known so that $\widehat{\Delta}(\cdot) = \mu(\cdot)$ and the choice $\widehat{\sigma}(\cdot) = \sigma(\cdot)$ is possible. In case of Gaussian i.i.d. ε_t independent of the X_t 's, \widehat{R}_{2K} would be an $\mathcal{N}(0, \widehat{R}_{1K}) = O_{\mathbb{P}}(\widehat{R}_{1K}^{1/2})$, which can be neglected with respect to \widehat{R}_{1K} when this variable diverges. Note also that the distribution of \widehat{R}_{3K} coincides with its Chi-square approximation for such $\widehat{\Delta}(\cdot)$, $\widehat{\sigma}(\cdot)$ and ε .

³Note that $\widehat{R}_K - c_K$ is a better misspecification indicator than \widehat{R}_K , which is affected by an additional systematic bias term c_K . Guerre and Lavergne (2005) proposed a different bias correction that makes asymptotic inference less accurate in finite sample, so that the Bootstrap is used.

let \mathcal{K}_T be a set of admissible degree K larger or equal to K_{\min} . Our data-driven choice of K is

$$\begin{aligned}\widehat{K}^\gamma &= \arg \max_{K \in \mathcal{K}_T} \left\{ \widehat{R}_K - c_K - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} \right\} \\ &= \arg \max_{K \in \mathcal{K}_T} \left\{ \widehat{R}_K - \widehat{R}_{K_{\min}} - (c_K - c_{K_{\min}}) - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} \right\} \text{ with } \gamma_T \geq 0.\end{aligned}\quad (2.5)$$

The introduction K_{\min} quantities in the penalty criterion reflects a preference for low degree as justified now from considerations on the null behavior of the retained $\widehat{R}_{\widehat{K}^\gamma}$.

As seen from Fan (1996) or Horowitz and Spokoiny (2001), finding an accurate approximation for the null distribution of a statistic which combines the \widehat{R}_K 's as $\widehat{R}_{\widehat{K}^\gamma}$ is difficult. A first distinctive feature is that the selection procedure (2.5) is flexible enough to limit the contribution of the statistics with high K by taking γ_T large enough. Indeed, a limit case is $\gamma_T = +\infty$, which gives that $\widehat{K}^\gamma = K_{\min}$. This continues to hold asymptotically provided γ_T diverges fast enough as shown in Theorem 1. Moreover, as detailed now, an accurate approximation of the distribution of $\widehat{R}_{\widehat{K}^\gamma}$ is a standard Chi-square. Since $\widehat{\Delta}(\cdot)$ asymptotically vanishes under H_0 , (2.3) shows that the null distribution of \widehat{R}_K is approximately the one of \widehat{R}_{3K} and then, neglecting the effect of the variance estimation, of

$$R_{3K} = \varepsilon' \Omega^{-1} \Psi_K (\Psi_K' \Omega^{-1} \Psi_K)^{-1} \Psi_K' \Omega^{-1} \varepsilon$$

where $\Omega^{1/2} = \text{Diag} [\sigma(X_1), \dots, \sigma(X_T)]$. In the i.i.d. case and according to the Berry-Esseen bound in Hart (1997, Theorem 7.2), the distribution of the vector

$$\Psi_K' \Omega^{-1} \varepsilon = \left[\sum_{t=1}^T \frac{\psi_k(X_t) \varepsilon_t}{\sigma^2(X_t)}, |k| \leq K \right]'$$

has a normal approximation up to a error $a(c_K)/T^{1/2}$ where $a(c_K)$ diverges with c_K . Therefore, the distribution of the Chi-squared statistic R_{3K} should be close to a Chi-square with c_K degree of freedom up to an error $a(c_K)/T^{1/2}$, which is smaller for moderate K .⁴ Hence the test uses a Chi-square critical value $z_\alpha = z_{\alpha, T}$ with

$$\mathbb{P} \left(\frac{\chi(c_{K_{\min}}) - c_{K_{\min}}}{\sqrt{2c_{K_{\min}}}} \geq z_\alpha \right) = \alpha$$

where $\chi(c)$ is a Chi-square with c degree of freedom and rejects H_0 if⁵

$$\frac{\widehat{R}^\gamma - c_{K_{\min}}}{\sqrt{2c_{K_{\min}}}} \geq z_\alpha \text{ where } \widehat{R}^\gamma = \widehat{R}_{\widehat{K}^\gamma}.\quad (2.6)$$

⁴This continues to hold in the dependent setup where the bound (B.9) in Appendix B gives a more complicate error term, which is $K^{2d}/T^{1/2}$ at best. A normal approximation would be affected with a bigger $K^{2d}/T^{1/2} + K^{-d/2}$ error term.

⁵A second distinctive feature of the selection procedure (2.5) is standardization with $c_{K_{\min}}$ in the critical region $\left\{ \widehat{R}^\gamma \geq c_{K_{\min}} + z_\alpha \sqrt{2c_{K_{\min}}} \right\}$, see (2.6). Since $\widehat{K}^\gamma = K_{\min}$ asymptotically, an alternative α -level critical region would use $c_{\widehat{K}^\gamma}$ in place of $c_{K_{\min}}$. But such a choice would asymptotically reduce power since $c_{\widehat{K}^\gamma} + z_\alpha \sqrt{2c_{\widehat{K}^\gamma}} \geq c_{K_{\min}} + z_\alpha \sqrt{2c_{K_{\min}}}$. This also contrasts with a maximum procedure which would use the test statistic $(\widehat{R}_{\widehat{K}^*} - c_{\widehat{K}^*})/\sqrt{2c_{\widehat{K}^*}} = \max_{K \in \mathcal{K}_T} (\widehat{R}_K - c_K)/\sqrt{2c_K}$ with a $c_{\widehat{K}^*}$ larger than $c_{K_{\min}}$. The simulation experiments of Guerre and Lavergne (2005) revealed that such a construction of the critical region (2.6) gives a test which improves on its adaptive rate-optimal competitors.

Consider now the power issue. The data-driven choice (2.5) of K combines the detection properties of each of the \widehat{R}_K 's. Indeed, since $c_K \geq c_{K_{\min}}$ for any K in \mathcal{K}_T , we have

$$\begin{aligned} \widehat{R}^\gamma - c_{K_{\min}} &\geq \widehat{R}_{\widehat{K}^\gamma} - c_{\widehat{K}^\gamma} = \max_{K \in \mathcal{K}_T} \left\{ \widehat{R}_K - c_K - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} \right\} + \gamma_T (2(c_{\widehat{K}^\gamma} - c_{K_{\min}}))^{1/2} \\ &\geq \max_{K \in \mathcal{K}_T} \left\{ \widehat{R}_K - c_K - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} \right\} \\ &\geq \widehat{R}_K - c_K - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} \end{aligned} \quad (2.7)$$

This gives the power lower bound

$$\begin{aligned} \mathbb{P} \left(\frac{\widehat{R}^\gamma - c_{K_{\min}}}{\sqrt{2c_{K_{\min}}}} \geq z_\alpha \right) &\geq \mathbb{P} \left(\widehat{R}_K - c_K - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2} - z_\alpha \sqrt{2c_{K_{\min}}} \geq 0 \right) \\ &= \mathbb{P} \left(\frac{\widehat{R}_K - c_K}{\sqrt{2c_K}} \geq \frac{z_\alpha \sqrt{2c_{K_{\min}}} + \gamma_T (2(c_K - c_{K_{\min}}))^{1/2}}{\sqrt{2c_K}} \right) \end{aligned} \quad (2.8)$$

which holds in particular for an optimal K which balances the bias with the penalty term. Taking $K = K_{\min}$ gives that

$$\mathbb{P} \left(\frac{\widehat{R}^\gamma - c_{K_{\min}}}{\sqrt{2c_{K_{\min}}}} \geq z_\alpha \right) \geq \mathbb{P} \left(\frac{\widehat{R}_{K_{\min}} - c_{K_{\min}}}{\sqrt{2c_{K_{\min}}}} \geq z_\alpha \right), \quad (2.9)$$

a power bound which shows that the test (2.6) improves on the one using the single statistic $\widehat{R}_{K_{\min}}$. As seen from (2.4) and (2.8), consistency holds as soon as there is a degree K in \mathcal{K}_T such that the misspecification measure $\sum_{t=1}^T \widehat{\Delta}^2(X_t) \mathbb{I}(X_t \in \Lambda) / \widehat{\sigma}^2(X_t)$ is asymptotically larger than the sum of $|\text{bias}_\mu(K)|$, $\gamma_T \sqrt{2(c_K - c_{K_{\min}})}$ and $z_\alpha \sqrt{2c_{K_{\min}}}$. Hence increasing too much γ_T should give a less powerful test. The form of the low penalty term in (2.5) is crucial to show adaptive rate-optimality, see Theorem 2. Theorem 3 shows that the test detects Pitman local alternatives with a rate arbitrarily close to the rate $T^{-1/2}$.

3 Main results

3.1 Main assumptions

Consider T observations (Y_t, X_t) with $Y_t = \mu(X_t) + \varepsilon_t$, $X_t = (Y_t, \dots, Y_{t-q}, Z_t) \in \mathbb{R}^d$, and where $\mu(\cdot)$ can depend upon T , in which case (Y_t, X_t) forms a triangular array (Y_{tT}, X_{tT}) . Let \mathcal{X}_t and $\overline{\mathcal{X}}_t$ denote the Borel field generated by $X_1, \varepsilon_1, \dots, X_t, \varepsilon_t$ and $X_t, \varepsilon_t, X_{t+1}, \varepsilon_{t+1}, \dots$ respectively. The α -mixing coefficients of $\{X_t, \varepsilon_t\}_{t \in \mathbb{N}^*}$ are

$$\alpha(n) = \sup_{t \in \mathbb{N}^*} \sup_{A \in \mathcal{X}_t, B \in \overline{\mathcal{X}}_{t+n}} |\mathbb{P}(A \cap B) - \mathbb{P}(A) \mathbb{P}(B)|, \quad n \in \mathbb{N}.$$

The next assumptions deal with the ε_t 's, the mixing coefficients and the parametric mean.

Assumption E Let \mathcal{F}_t be the Borel field generated by $(X_1, \varepsilon_0), \dots, (X_t, \varepsilon_{t-1})$. The variables $\{\varepsilon_t\}_{t \in \mathbb{N}}$ are martingale difference with $\mathbb{E}[\varepsilon_t | \mathcal{F}_t] = 0$, $\mathbb{E}[\varepsilon_t^2 | \mathcal{F}_t] = \sigma^2(X_{t-1})$ and $\sup_{t \in \mathbb{N}} \mathbb{E}[\varepsilon_t^8 | \mathcal{F}_t] < \infty$ a.s.. The standard deviation function, $\sigma(\cdot) = \text{Var}[\varepsilon_t | X_t = \cdot]$, is continuous and bounded away from 0 on \mathbb{R}^d .

Assumption X The process $\{X_t, \varepsilon_t\}_{t \in \mathbb{N}^*}$ on $\mathbb{R}^d \times \mathbb{R}$ is stationary, with

- i. $\alpha(n) \leq An^{-1-a}$ for some constant $A, a > 0$.
- ii. The variable X_t has a density $f(\cdot)$ with respect to the Lebesgue measure on \mathbb{R}^d . The density $f(\cdot)$ is bounded away from 0 and infinity.

Assumption M The parameter set Θ is a subset of \mathbb{R}^p and

- i. The regression function $m(x; \theta)$ is twice continuously differentiable with respect to θ . The gradient $m^{(1)}(x; \theta)$ and Hessian matrix $m^{(2)}(x; \theta)$ are bounded over $\Lambda \times \Theta$.
- ii. For any sequence of regression functions $\mu_T(\cdot)$ with $\mathbb{E}\mu_T^2(X_t) < \infty$, there exists a sequence of parameter θ_T in Θ such that $T^{1/2}(\widehat{\theta}_T - \theta_T) = O_{\mathbb{P}}(1)$, with $\theta_T = \theta$ if $\mu_T(\cdot) = m(\cdot; \theta)$ for some θ in Θ .

Assumption E ensures that the sums $\sum_{t=1}^T \psi_k(X_t) \varepsilon_t / \sigma^2(X_t)$ are martingales which are asymptotically normal under Assumption X-(i). The polynomial mixing rate of X-(i) is a minimal rate to achieve $T^{1/2}$ -consistency in the weak Law of Large Numbers for the empirical mean $T^{-1} \Psi'_K \Omega^{-1} \Psi_K$. Under Assumption X-(ii), the limit of $T^{-1} \Psi'_K \Omega^{-1} \Psi_K$ has an inverse. Mixing conditions for Markovian (Y_t, X_t) as in Assumption X-(i) can be derived using a drift condition, see Fan and Yao (2003, Theorem 2.4) and the references therein. When $\widehat{\theta}_T = \arg \min_{\theta \in \Theta} \sum_{t=1}^T (Y_t - m(X_t; \theta))^2$, the sequence θ_T in Assumption M-(ii) is the pseudo-true value $\arg \min_{\theta \in \Theta} \mathbb{E}(\mu_T(X_t) - m(X_t; \theta))^2$, which is uniquely defined under identification of the parametric regression model, see Domowitz and White (1982). Assumption M-(i) then ensures that $\widehat{\Delta}(\cdot) = \mu_T(\cdot) - m(\cdot; \widehat{\theta}_T)$ is close to $\Delta(\cdot) = \mu_T(\cdot) - m(\cdot; \theta_T)$ over Λ up to an $O_{\mathbb{P}}(T^{-1/2})$ term.

Let us now turn to the construction of the test. The first assumption specifies a set of admissible degrees \mathcal{K}_T in the spirit of the dyadic bandwidth set of Horowitz and Spokoiny (2001).

Assumption K Let a be as in Assumption X. Set $K_{\max} = 2^{J_{\max}} = O(T^{C_1/d})$ for some C_1 in $(0, \frac{3}{4} \frac{1+a}{5+3a})$, $K_{\min} = 2^{J_{\min}} \rightarrow \infty$ with $K_{\min}^d = O(\ln^{C_2} T)$ for $C_2 > 0$, where $J_{\min} \leq J_{\max}$ are integer numbers. The set of admissible degrees \mathcal{K}_T is dyadic, that is

$$\mathcal{K}_T = \{K = 2^J, J = J_{\min}, J_{\min} + 1, \dots, J_{\max}\}. \quad (3.1)$$

Note that (3.1) and the polynomial divergence rate of K_{\max} imply that $\text{Card}\mathcal{K}_T$ is of exact order $\ln T$. Such a restriction is helpful to show that $\widehat{K}^\gamma = K_{\min}$ asymptotically under the null, but also have some practical justifications. Indeed, achieving a small $\mathbb{P}\left(\widehat{K}^\gamma \neq K_{\min}\right)$ is an important condition to get an accurate size. Since $\widehat{R}_K - \widehat{R}_{K_{\min}} - (c_K - c_{K_{\min}}) - \gamma_T (2(c_K - c_{K_{\min}}))^{1/2}$ vanishes if and only if $K = K_{\min}$, (2.5) yields that $\widehat{K}^\gamma \neq K_{\min}$ if and only if one of these penalized statistics is strictly positive for a $K \neq K_{\min}$, or equivalently

$$\max_{K \in \mathcal{K}_T \setminus \{K_{\min}\}} \frac{\widehat{R}_K - \widehat{R}_{K_{\min}} - (c_K - c_{K_{\min}})}{(2(c_K - c_{K_{\min}}))^{1/2}} > \gamma_T .$$

Hence

$$\mathbb{P}\left(\widehat{K}^\gamma \neq K_{\min}\right) = \mathbb{P}\left(\max_{K \in \mathcal{K}_T \setminus \{K_{\min}\}} \frac{\widehat{R}_K - \widehat{R}_{K_{\min}} - (c_K - c_{K_{\min}})}{(2(c_K - c_{K_{\min}}))^{1/2}} \geq \gamma_T\right), \quad (3.2)$$

so that $\mathbb{P}\left(\widehat{K}^\gamma \neq K_{\min}\right)$ increases with \mathcal{K}_T and decreases with the penalty sequence γ_T . Therefore, using a parsimonious \mathcal{K}_T can improve the size accuracy of the test. On the other hand, a dyadic \mathcal{K}_T as in Assumption K contains sequences with any arbitrary order between $\ln^{C_2} T$ and T^{C_1} which is sufficient for adaptive rate-optimality. The constant C_1 of Assumption K must be smaller than $\frac{3}{4} \frac{1+a}{5+3a}$ where a comes from Assumption X-(i), $\alpha(n) = O(n^{-1-a})$. This gives a K_{\max} of order $T^{1/(4d)}$ at best, while, in the i.i.d. set-up, Hong and White (1995) allowed for a better order $T^{1/(3d)}$ when using a single series statistic to base the test.

Let us now turn to variance estimation. The next condition allows to approximate $T^{-1}\Psi'_K \widehat{\Omega}^{-1} \Psi_K$ with $T^{-1}\Psi'_K \Omega^{-1} \Psi_K$ for degrees K depending on the sample size T , as in Assumption K.

Assumption V Let $K_{\max} = 2^{J_{\max}} = \max\{K; K \in \mathcal{K}_T\}$. Then, for the considered sequence of regression models $Y_t = \mu_T(X_t) + \varepsilon_t$, $\sup_{x \in \Lambda} |\widehat{\sigma}(x) - \sigma(x)| = O_{\mathbb{P}}(v_T)$ and, for some integer $\ell > d/2$ and all (ℓ_1, \dots, ℓ_d) with $\ell_1 + \dots + \ell_d = \ell$, $\sup_{x \in \Lambda} \left| \frac{\partial^\ell \widehat{\sigma}(x)}{\partial^{\ell_1} x_1 \dots \partial^{\ell_d} x_d} \right| = O_{\mathbb{P}}(v_T)$, where $v_T = o\left(K_{\max}^{-3d/2} / \ln T\right)$ and $\liminf_{T \rightarrow \infty} T^{1/2} v_T > 0$.

Assumption V requires consistency of $\widehat{\sigma}(\cdot)$ under the null and the alternative. Convergence of $\widehat{\sigma}(\cdot)$ with the rate v_T requires that $\mu_T(\cdot)$ and $\sigma(\cdot)$ satisfies a minimal smoothness condition. As seen from Guerre and Lavergne (2002), consistency is not necessary under the alternative but can be useful to get a powerful test. Under homoskedasticity, a simple choice of $\widehat{\sigma}(\cdot)$ is a constant difference-based estimator, in which case Assumption V holds with a best possible $v_T = T^{-1/2}$ so that $K_{\max} = o\left(T^{1/(3d)} \ln^{2/(3d)} T\right)$. The heteroskedastic case requires nonparametric variance estimation, as kernel, sieves, series expansion, see, among others, Guerre and Lavergne (2002, 2005) and Horowitz and Spokoiny (2001). The rate v_T is then the consistency rate for the ℓ -th partial derivatives, which restricts the divergence rate of K_{\max} .

3.2 Asymptotic behavior under the null

As discussed following (3.2) and (2.8), a fast divergence rate for γ_T is useful to achieve an accurate size under the null but may negatively affect its power properties. Therefore, an important issue is to find a minimal

divergence rate for γ_T ensuring that the test is asymptotically of level α , or equivalently that $\mathbb{P}(\widehat{K}^\gamma \neq K_{\min})$ asymptotically vanishes under H_0 . The Bonferroni inequality gives, in (3.2),

$$\mathbb{P}(\widehat{K}^\gamma \neq K_{\min}) \leq \sum_{K \in \mathcal{K}_T \setminus \{K_{\min}\}} \mathbb{P} \left(\frac{\widehat{R}_K - \widehat{R}_{K_{\min}} - (c_K - c_{K_{\min}})}{(2(c_K - c_{K_{\min}}))^{1/2}} > \gamma_T \right), \quad (3.3)$$

and showing that the last sum asymptotically vanishes for small γ_T necessitates precise uniform bounds for these probabilities, so that simple Chebychev-type inequalities may not be sufficient. Better Gaussian-type bounds in spirit of the Mill's ratio inequality $\mathbb{P}(\mathcal{N}(0, 1) \geq \gamma) \leq \exp(-\gamma^2/2)/(\sqrt{2\pi}\gamma)$ are derived in Lemma A.3 of Appendix A. Since the exact order of $\text{Card}\mathcal{K}_T$ is $\ln T$, the next theorem ensures that the asymptotic size of the test is α provided that the penalty sequence γ_T diverges faster than $(\ln \ln T)^{1/2}$.

Theorem 1 *Consider that the null hypothesis H_0 is true and assume that Assumptions E, K, M, V and X hold. Then, if γ_T diverges with*

$$\gamma_T \geq (1 + \epsilon)\sqrt{2 \ln \text{Card}\mathcal{K}_T}, \text{ for some } \epsilon > 0, \quad (3.4)$$

$\lim_{T \rightarrow +\infty} \mathbb{P}(\widehat{K}^\gamma = K_{\min}) = 1$ and the test (2.6) is asymptotically of level α .

The minimal divergence rate $(\ln \ln T)^{1/2}$ ensuring that the test is asymptotically of level α is surprisingly low compared to the penalty term of order $\ln T$ used in the BIC criterion. Such improvement comes from the Gaussian-type bounds used for the tails of the standardized $\widehat{R}_K - \widehat{R}_{K_{\min}}$'s. Indeed, this gives, up to remainder terms, a bound $\text{Card}\mathcal{K}_T \exp(-\gamma_T^2/2)/(\sqrt{2\pi}\gamma_T)$ in (3.3), which asymptotically vanishes provided that (3.4) holds. On the other hand, such a low rate is in line with previous findings for rate-optimal adaptive testing. Indeed, (3.2) shows that suitable γ_T should resemble to the critical values of a maximum tests as Fan (1996), who found critical values with a typical rate of $(2 \ln \ln T)^{1/2}$. This suggests that our minimal rate condition (3.4) cannot be improved.

Another condition for Theorem 1 to hold is that K_{\min} diverges with the sample size, see Assumption K. This is used to neglect the parametric estimation error $T^{1/2}(\widehat{\theta}_T - \theta)$ in the Chi-square approximation of the distribution of $\widehat{R}_{K_{\min}}$. Accounting for such an effect would allow to consider a fixed K_{\min} , see e.g. Hart (1997, Section 8.3.1).

3.3 Detection of small alternatives

As discussed following Equation (2.8), the detection properties of the test depend upon a bias term from (2.4). Establishing formal adaptive rate-optimality of the test necessitates to bound this bias. The current mathematical approach to do so makes use of some smoothness restrictions. We consider here Hölder smoothness classes $\mathcal{C}(L, s)$ that we introduce now. Define the departure from the null as,

$$\Delta_{\mu, T}(\cdot) = \mu(\cdot) - m(\cdot; \theta_T),$$

with a θ_T as in Assumption M. We restrict to departures $\Delta(\cdot)$ with a restriction to Λ which admits a (2λ) -periodic extension. Consider first the case $s \in (0, 1]$, for which

$$\mathcal{C}(L, s) = \left\{ \Delta(\cdot) : \sup_{x, x' \in \Lambda} \frac{|\Delta(x) - \Delta(x')|}{\|x - x'\|^s} \leq L \right\}.$$

For real $s > 0$, let $\lfloor s \rfloor$ be the lower integer part of s , that is the unique integer number satisfying $\lfloor s \rfloor < s \leq \lfloor s \rfloor + 1$, so that $s - \lfloor s \rfloor$ is in $(0, 1]$ with $s - \lfloor s \rfloor = s$ for $s \in (0, 1]$. For any $s > 0$, the smoothness class $\mathcal{C}(L, s)$ is defined as

$$\mathcal{C}(L, s) = \{ \Delta(\cdot) : \text{the } \lfloor s \rfloor \text{th partial derivatives of } \Delta(\cdot) \text{ are in } \mathcal{C}(L, s - \lfloor s \rfloor) \}.$$

Hence the smoothness class $\mathcal{C}(L, s)$ is defined for all $s > 0$ and $L > 0$. Lemma 1 in the Proof Section gives, for the bias term of (2.4), the following bound

$$|\text{bias}_\mu(K)| \leq O_{\mathbb{P}} \left[T^{1/2} K^{-s} \mathbb{E}^{1/2} \left(\frac{\Delta_{\mu, T}(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)} \right)^2 + TK^{-2s} \right],$$

for any $\Delta_{\mu, T}(\cdot)$ in $\mathcal{C}(L, s)$ and any K . This gives, for small alternatives which are the harder to detect,

$$|\text{bias}_\mu(K)| \leq O_{\mathbb{P}} (TK^{-2s}) \text{ provided } \mathbb{E}^{1/2} \left(\frac{\Delta_{\mu, T}(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)} \right)^2 = O(K^{-s}). \quad (3.5)$$

Our minimax adaptive framework evaluates tests uniformly over alternatives at distance ρ from the null, that is in

$$H_1(\rho; L, s) = \left\{ \mu(\cdot) = m(\cdot; \theta_T) + \Delta_{\mu, T}(\cdot); \Delta_{\mu, T}(\cdot) \in \mathcal{C}(L, s), \mathbb{E} \left(\frac{\Delta_{\mu, T}(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)} \right)^2 \geq \rho^2 \right\},$$

with unknown smoothness index (L, s) . Such alternatives allow for a general shape of $\Delta_{\mu, T}(\cdot)$ with narrow peaks and valleys that may depend upon on T , see Horowitz and Spokoiny (2001). As pointed out in Guerre and Lavergne (2005), uniform consistency over $H_1(\tilde{\rho}_T; L, s)$ is equivalent to consistency against any sequence

$$\mu_T(\cdot) = m(\cdot; \theta_T) + \Delta_T(\cdot), \text{ where } \Delta_T(\cdot) = \Delta_{\mu_T, T}(\cdot),$$

in $H_1(\tilde{\rho}_T; L, s)$ as considered here. A crucial issue is the choice of a suitable asymptotically vanishing rate $\tilde{\rho}_T$. Indeed, some of the alternatives of $H_1(\tilde{\rho}_T; L, s)$ will not be detected by any tests if $\tilde{\rho}_T$ goes to 0 with a too fast rate. On the other hand, detection can become straightforward if $H_1(\tilde{\rho}_T; L, s)$ remains far from the null. Hence a good candidate $\tilde{\rho}_T$ to evaluate a test is a frontier rate which separates these two extreme situations. In the adaptive approach, such a rate depends upon the unknown smoothness index s and Spokoiny (1996) has shown that the optimal adaptive rate is⁶

$$\rho_T = \rho_T(s) = \left(\frac{\sqrt{\ln \ln T}}{T} \right)^{\frac{2s}{4s+d}},$$

⁶Spokoiny (1996) studied the continuous time white noise model (CTWN) $Y_n(t) = m(t)dt + \frac{\sigma}{\sqrt{n}}dW(t)$, $t \in [0, 1]$, where $\{W(t)\}_{t \in [0, 1]}$ is a standard Brownian motion. Although this model is mainly of theoretical interest, results established for the CTWN model extend to more common models through model equivalence, see Brown and Low (1996).

which is slower than the parametric rate $T^{-1/2}$. Guerre and Lavergne (2002) derived an optimal rate for a known smoothness index s which improves ρ_T from the $(\ln \ln T)^{1/2}$ factor, so that the price to pay for rate adaptation is moderate. As well-known, the rate ρ_T decreases faster than the nonparametric estimation rate $T^{-s/(2s+d)}$. The adaptive rate-optimality of our test is stated in the next result.

Theorem 2 *Consider a sequence of alternatives*

$$\mu_T(\cdot) = m(\cdot; \theta_T) + \Delta_T(\cdot) \text{ in } H_1(C_3, \rho_T; L, s) \text{ for some unknown } s \text{ and } L,$$

with $s \geq d(2/C_1 - 1)/4$, $L > 0$, $C_3 > 0$ and $\sup_{x \in \Lambda} |\Delta_T(x)| = O\left[\mathbb{E}^{1/2}(\Delta_T^2(X_t) \mathbb{I}(X_t \in \Lambda)) / \sigma^2(X_t)\right]$. Assume that Assumptions E, K, M, V hold. Then, if γ_T is of exact order $(\ln \ln T)^{1/2}$ and provided C_3 is taken large enough, the test is consistent, that is $\lim_{T \rightarrow \infty} \mathbb{P}\left(\left(\widehat{R}^\gamma - c_{K_{\min}}\right) / \sqrt{2c_{K_{\min}}}\right) \geq z_\alpha\right) = 1$.

The proof of Theorem 2 builds on the lower power bound (2.8) and on the bias variance decomposition (2.4). In view of the bias order (3.5) for small alternatives, an optimal choice of K in (2.8) is such that the order of the penalty term $\gamma_T K^{d/2}$ is proportional to TK^{-2s} , that is for

$$K_* = K_*(s) = 2^{\left[\frac{2}{4s+d} \frac{\ln(T/\gamma_T)}{\ln 2}\right]} \propto \left(\frac{T}{\gamma_T}\right)^{\frac{2}{4s+d}}, \quad (3.6)$$

where $[\cdot]$ is the integer part. Such K_* detects alternatives within the bias order divided by the sample size, $K_*^{-s} \propto (\gamma_T/T)^{2s/(4s+d)}$, which coincides with the optimal adaptive order ρ_T provided γ_T has the smallest possible order $(\ln \ln T)^{1/2}$ compatible with Theorem 1. Note that, under Assumption K, K_* is in \mathcal{K}_T provided $s \geq d(2/C_1 - 1)/4$, which implies that $s > 7d/4$.

Because adaptation means detection over various smoothness classes $\mathcal{C}(L, s)$, it is crucial that the test combines various several statistics, as seen from the optimal K_* in (3.6), which depends on the smoothing index s . Therefore, tests that use a single statistic \widehat{R}_K generally fail to be rate-optimal adaptive. A more specific property of the test (2.6) is detection of small local alternatives.

Theorem 3 *Consider a sequence of local alternatives $\mu_T(\cdot)$ satisfying*

$$\mu_T(X_t) = m(X_t; \theta_T) + r_T \Delta_{0T}(X_t) \text{ with } \Delta_{0T}(\cdot) \text{ in } \mathcal{C}(L, s) \text{ for some unknown } s, L > 0,$$

$s \geq d(2/C_1 - 1)$, and

$$\mathbb{E}\left(\frac{\Delta_{0T}(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)}\right)^2 \geq 1, \quad \sup_{x \in \Lambda} |\Delta_{0T}(x)| = O(1).$$

Then, under Assumptions E, K, M, V and X, the test is consistent provided $1/r_T = o\left(\sqrt{T/K_{\min}^{d/2}}\right)$.

Since K_{\min} can diverge very slowly, the rate r_T can be arbitrarily close to the parametric detection rate $1/T^{1/2}$. This slightly improves on Horowitz and Spokoiny (2001) who achieved a rate $(\ln \ln T)^{1/2}/T^{1/2}$. A

key argument there is that the local alternatives of Theorem 3 are asymptotically very smooth, since the departure from the null $r_T \Delta_{0T}(\cdot)$ are in $\mathcal{C}(Lr_T, s)$, with a Lipschitz constant Lr_T which goes to 0. Hence these alternatives differ from the general ones in Theorem 2, and are typically detected by trigonometric series with low degree as K_{\min} , so that (2.9) yields consistency of the test. On the other hand, using the single statistic $\widehat{R}_{K_{\min}}$ would give a test that is not consistent against the alternatives of Theorem 2, so that combining several statistics as in our procedure is crucial to achieve these opposite kinds of detection properties.

4 Simulation experiments

In this section we study the size and the power properties of the proposed procedure when testing for a null of linearity in the context of a Markov process of order 1. The resulting test is compared with the one developed by Hamilton (2001) to detect nonlinearity. First, to examine the size properties, we use the AR(1)

$$Y_t = \rho Y_{t-1} + \epsilon_t.$$

Three distributions are considered for the error term: standard normal, standardized student with five degrees of freedom, and a centered and standardized exponential. To examine the sensitivity of the tests to temporal dependence, we consider various values of the autoregressive parameter ρ , namely $\rho = 0, .25, .50, .75$. To implement our test, we choose the interval (Λ in Section 2) for projecting the covariate Y_{t-1} onto the trigonometric expansion to be equal to 2 times the standard error of Y_t under the null. This corresponds to approximately 95 % of the observations. The set \mathcal{K}_T is equal to $\{1, 2, 4, 8, 16\}$. The asymptotic critical value is given by $\sqrt{2} * \chi_{.05}(1) + 1$, where $\chi_{.05}(1)$ is the critical value at 5 % of a Chi-square with one degree of freedom. We study the small sample properties of the test for various values of the penalty parameter γ_T . We fix γ_T equal to $c\sqrt{2 \ln \text{Card} \mathcal{K}_T}$ where we set $c = 2, 3, 5$. The parameters are estimated by OLS. The sample size is set to 200 and the number of simulations is equal to 10,000.

The simulations results for the size are encouraging. For $c = 2$ the test slightly overrejects in all cases. However, for $c = 3, 5$, the size is accurate whatever the distribution, persistence and number of observations considered. The LM test developed by Hamilton (2001) shares these good size properties.

(Insert Table 1 around here)

To study the effect on power of the penalty sequence γ_T , two alternative specifications of the linear autoregressive process are examined. The first specification is a threshold autoregressive model defined as:

$$Y_t = \rho_1 Y_{t-1} \mathbb{I}_{\{Y_{t-1} > 0\}} + \rho_2 Y_{t-1} \mathbb{I}_{\{Y_{t-1} < 0\}} + \epsilon_t$$

where ϵ_t is i.i.d. $N(0, 1)$.⁷ This representation contains two regimes delimited by a threshold equal to zero. When Y_{t-1} is greater than zero, the dynamic dependence is controlled by the parameter ρ_1 . In the case where it is inferior to zero, the dynamic depends on the parameter ρ_2 . Under the null of linearity $\rho_1 = \rho_2$. The distance from the null is a function of the absolute value of the difference between ρ_1 and ρ_2 . To see this, we can rewrite the threshold autoregressive model as follows:

$$Y_t = \rho_1 Y_{t-1} + (\rho_2 - \rho_1) Y_{t-1} \mathbb{I}_{\{Y_{t-1} < 0\}} + \epsilon_t.$$

Thus, under the null, $\mu(X_t) = \rho_1 Y_{t-1}$ while the nonlinear alternative is

$$\mu(X_t) = \rho_1 Y_{t-1} + \delta Y_{t-1} \mathbb{I}(Y_{t-1} < 0), \text{ where } \delta = \rho_2 - \rho_1.$$

To examine the sensitivity of the tests to temporal dependence, we consider various types of dependence for the process Y_t . We run the following experiments: (1) $\rho_1 = 0$ and $\rho_2 = .25, .50, .75$, (2) $\rho_1 = .25$ and $\rho_2 = .50, .75, -.50$, (3) $\rho_1 = .50$ and $\rho_2 = .25, 0, -.25$ and (4) $\rho_1 = .75$ and $\rho_2 = .50, .25, 0$. The values of ρ_2 under the alternative are chosen such that the parameter (δ) which governs the distance from the null is equal to .25, .50 and .75, respectively. Table 2 reports the power results. Our test is more powerful than Hamilton's for all cases. Our power gains increase with the degree of temporal dependence and the distance of the alternative from the null. The difference in the rejection rate can be as high as 38%.

(Insert Table 2 around here)

The second experiment corresponds to an alternative for which the data-driven optimal test is specially designed. The alternative models have the following form:

$$Y_t = \rho Y_{t-1} + \frac{\rho}{\tau} f(Y_{t-1}/\tau) + \epsilon_t \quad (4.7)$$

where $f(y/\tau) = (1/\sqrt{2\pi\sigma^2}) \times \exp(-\frac{1}{2\sigma^2}(y/\tau)^2)$, $\sigma^2 = 1/(1 - \rho^2)$ and ϵ_t is i.i.d. $N(0, 1)$. Figure 1 shows the function $f(\cdot)$ for $\tau = 1, .5$ and $.25$, $\rho = .5$ and values of Y_t between -10 and 10 . The function $f(\cdot)$ is symmetric around zero and more concentrated for smaller values of τ . The function is bounded between zero and one, with $f(0) = 1$ and $\lim_{x \rightarrow \pm\infty} f(x) = 0$. We can easily show that the alternative (4.7) respects the drift condition of Fan and Yao (2003, Theorem 2.4) for geometric ergodicity. This alternative is then compatible with the assumptions assumed in this paper.

(Insert Table 3 around here)

We examine the sensitivity of the tests to the narrowness of the peak and temporal dependence. We consider the parameter values $\tau = 25, .5, .75$ and $\rho = .25, .5, .75$. Table 3 shows the results of the experiment.

⁷Results for the normal distribution are only reported here because the results for the two other distributions are very similar. Of course, those results can be obtained upon request.

For $\tau = 1$, Hamilton's test is close to the nominal size. For 200 observations, our test rejects at a rate of 16 % for $\rho = .25$ and 56 % for $\rho = .75$. For $\tau = .5$, our test also clearly dominates the test proposed by Hamilton for all cases. For a narrow peak ($\tau = .25$), the rejection rate of both tests is quite similar. The better performance of the Hamilton test for this alternative, compared to the one with a wider peak is probably due to the specification of the variance-covariance function of the random field underlying the test statistic, see Hamilton (2001) for further details on the construction of this test.

5 Concluding remarks

This paper proposes a new adaptive rate-optimal specification test for time series. As in the maximum approach of Fan (1996) or Horowitz and Spokoiny (2001), the test combines several statistics to achieve adaptive rate-optimality. More specifically, the test builds on series regression Chi-square statistics with increasing orders. A data-driven selection procedure, in the spirit of Guerre and Lavergne (2005), uses a penalty term proportional to the square-root of the number of Fourier coefficients to choose the test statistic. Under the null, the retained statistic is, with high probability, a statistic with a distribution close to a Chi-square. Therefore, standard Chi-square critical values can be used, allowing for a better control of the size of the test. This contrasts with the maximum approach, where using a null limit distribution performs poorly as noted in Fan (1996), or is out of reach, as in Horowitz and Spokoiny (2001). Hence, the maximum approach necessitates the use of simulated critical values, limiting the scope of applications to time series models than can be easily simulated. A simulation experiment confirms the good level properties of the proposed test, which shows interesting power improvements compared to a simpler test using a single statistic as Hamilton (2001). We also examine the power of the test which is adaptive rate-optimal and detects local alternatives approaching the null at a faster rate than Horowitz and Spokoiny (2001). The simulation experiment shows that the choice of the penalty term has a moderate impact on the power. This positively illustrates the interest of our approach, which builds on the fact that the combination mechanism inherent to adaptive testing can also be designed to achieve a level close to the nominal size.

While our results are stated for Fourier series methods, our approach applies as well to wavelets or polynomial series regression. As noted in Guerre and Lavergne (2005), the series construction of the test statistic can be easily modified to cope with additive alternatives which are not affected by the curse of dimensionality. Obtaining an accurate size in case of kernel or local polynomial methods is theoretically feasible. The scope of applications of the new data-driven selection procedure can also be extended as discussed in Hart (1997) for earlier adaptive procedures or as in Tjøstheim (1994) and Fan and Yao (2003) in the time series context, as well as to many other specification hypotheses of econometric interest.

6 Proofs of main results

The proofs are organized as follows. Important intermediate results and proofs of the main statements are given in Section 6. Proofs of auxiliary results are gathered in Appendices A and B. We now introduce some notations and conventions. All functions can be set to 0 outside Λ without loss of generality. We set $\sum_{t=0}^{-1} = \sum_{t=T+1}^T = 0$. The symbol $a_T \asymp b_T$ means that the two sequences a_T, b_T with same sign are such that $c|a_T| \leq |b_T| \leq C|a_T|$ for some $0 < c \leq C < \infty$ and $T \geq 1$. Constants are denoted by the generic letter C and vary from line to line.

For notation convenience, we re-index the trigonometric functions (2.1) as $\{\psi_k(\cdot)\}_{k \in \mathbb{N}^*}$ and set $c_K = \kappa$. We assume that the new ordering is such that $\Psi_K = [\psi_1, \dots, \psi_\kappa]$, and uses the notation Ψ_κ for Ψ_K . Here $\psi_k, k \in \mathbb{N}^*$, is a column vector with $\psi_k = [\psi_k(X_1), \dots, \psi_k(X_T)]' \in \mathbb{R}^T$. Therefore Ψ_κ is a $T \times \kappa$ matrix and $\kappa \asymp K^d$. With little abuse of notation, \mathcal{K}_T denotes both the set of admissible K or κ with κ between $\kappa_{\min} \asymp 2^{J_{\min}^d}$ and $\kappa_{\max} \asymp 2^{J_{\max}^d}$. $\widehat{\kappa}^\gamma$ corresponds to \widehat{K}^γ . The variance estimation rate in Assumption V is such that $v_T = o(\kappa_{\max}^{-3/2} / \ln T)$.

Let $\|\cdot\|$ be the Euclidean norm of \mathbb{R}^T or \mathbb{R}^κ , that is if $u = [u_1, \dots, u_\kappa]' \in \mathbb{R}^\kappa$, $\|u\| = (\sum_{k=1}^\kappa u_k^2)^{1/2} = (u'u)^{1/2}$. If $m = [m(X_1), \dots, m(X_T)]'$ where $m(\cdot)$ maps \mathbb{R}^d to \mathbb{R} , $\|m\| \leq T^{1/2} \sup_{x \in \mathbb{R}^d} |m(x)|$. Under Assumption E, $\|\varepsilon\| = O_{\mathbb{P}}(T^{1/2})$. For a $\kappa \times \kappa$ matrix $\Sigma = [\Sigma_{k\ell}]_{1 \leq k, \ell \leq \kappa}$, $\|\Sigma\|$ is the spectral radius $\|\Sigma\| = \sup_{u \neq 0 \in \mathbb{R}^\kappa} \|\Sigma u\| / \|u\|$. Recall that $\|\Sigma u\| \leq \|\Sigma\| \|u\|$, $|u_1' \Sigma u_2| \leq \|\Sigma\| \|u_1\| \|u_2\|$. It follows that the entries of Σu are bounded by $\kappa^{1/2} \|\Sigma\| \max_{1 \leq k \leq \kappa} |u_k|$. If Σ is a symmetric matrix, $\|\Sigma\| = \sup_{\|u\|=1} |u' \Sigma u|$ is the largest eigenvalue in absolute value of Σ . Since $\widehat{\Omega}^{-1/2} \Psi_\kappa \left(\Psi_\kappa' \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi_\kappa' \widehat{\Omega}^{-1/2}$ is the orthogonal projection on the space spanned by the columns of $\widehat{\Omega}^{-1/2} \Psi_\kappa$, we have

$$\left| u' \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi_\kappa' \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi_\kappa' \widehat{\Omega}^{-1} u \right| \leq \left\| \widehat{\Omega}^{-1/2} u \right\|^2 \quad \text{and} \quad \left\| \widehat{\Omega}^{-1/2} \Psi_\kappa \left(\Psi_\kappa' \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi_\kappa' \widehat{\Omega}^{-1/2} \right\| \leq 1.$$

In what follows, we bound variance of sums using the Wolkonski-Rozanov inequality (see Fan and Yao (2003), Proposition 2.5-ii) which states that

$$|\text{Cov}(g_1(X_t), g_2(X_{t-n}))| \leq 4\alpha(n) \sup_{x \in \mathbb{R}^d} |g_1(x)| \sup_{x \in \mathbb{R}^d} |g_2(x)|$$

for any real-valued bounded $g_1(\cdot)$ and $g_2(\cdot)$. This gives

$$\text{Var} \left(\frac{1}{T} \sum_{t=1}^T g(X_t) \right) = \frac{1}{T} \left(\text{Var}(g(X_1)) + 2 \sum_{n=1}^T \frac{T-n}{T} \text{Cov}(g(X_1), g(X_{n-1})) \right) \leq \frac{8}{T} \sup_{x \in \mathbb{R}^d} |g(x)|^2 \sum_{n=0}^{\infty} \alpha(n). \quad (6.1)$$

6.1 Estimation errors

We consider first the parametric and variance estimation errors, induced by $\widehat{\theta}_T - \theta_T$ and $\widehat{\sigma}(\cdot) - \sigma(\cdot)$ respectively. For $\Delta_T(\cdot) = \mu_T(\cdot) - m(x; \theta_T)$, set $U = \Delta_T + \varepsilon$, and let $\Omega^{1/2}$ be the $T \times T$ diagonal matrix with entries $\sigma(X_t)$. Set

$$\Sigma_\kappa = \mathbb{E} \left[\frac{\Psi_\kappa'(X_t) \Psi_\kappa(X_t)}{\sigma^2(X_t)} \right] = \left[\mathbb{E} \left(\frac{\psi_k(X_t) \psi_\ell(X_t)}{\sigma^2(X_t)} \right) \right]_{1 \leq k, \ell \leq \kappa}, \quad \widehat{\Sigma}_\kappa = \widehat{\Sigma}_\kappa(\widehat{\Omega}) = \left[\frac{\overline{\psi_k(X) \psi_\ell(X)}}{\widehat{\sigma}^2(X)} \right]_{1 \leq k, \ell \leq \kappa}, \quad (6.2)$$

where

$$\frac{\overline{\psi_k(X) \psi_\ell(X)}}{\widehat{\sigma}^2(X)} = \frac{1}{T} \sum_{t=1}^T \frac{\psi_k(X_t) \psi_\ell(X_t)}{\widehat{\sigma}^2(X_t)}, \quad \text{so that } T \widehat{\Sigma}_\kappa = \Psi_\kappa' \widehat{\Omega}^{-1} \Psi_\kappa \quad \text{and} \quad \widehat{R}_\kappa = U' \widehat{\Omega}^{-1} \Psi_\kappa \left(T \widehat{\Sigma}_\kappa \right)^{-1} \Psi_\kappa' \widehat{\Omega}^{-1} U.$$

Proposition 1 Consider a departure from the null such that $\sup_{x \in \Lambda} |\Delta_T(x)| = O\left[\mathbb{E}^{1/2}(\Delta_T(X_t)/\sigma(X_t))^2\right]$. Under Assumptions E, M, V, X and if $\kappa_{\min} \rightarrow \infty$, $\kappa_{\max} = O(T^{1/3}/\ln^2 T)$, we have

$$\begin{aligned} & \max_{\kappa \in \mathcal{K}_T} \frac{\left| \widehat{R}_\kappa - \varepsilon' \Omega^{-1} \Psi_\kappa (T\Sigma_\kappa)^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon - 2\Delta_T \widehat{\Omega}^{-1} \Psi_\kappa (T\widehat{\Sigma}_\kappa)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon - \Delta_T \widehat{\Omega}^{-1} \Psi_\kappa (T\widehat{\Sigma}_\kappa)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \Delta_T \right|}{\kappa^{1/2}} \\ &= O_{\mathbb{P}} \left(\frac{T^{1/2}}{\kappa_{\min}^{1/2}} \mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)} \right)^2 \right). \end{aligned}$$

Proof of Proposition 1: see Appendix A.

6.2 Proof of Theorem 1

The next Proposition is the key tool to establish Theorem 1.

Proposition 2 Assume that H_0 holds, that is $\Delta_T(\cdot) = 0$. Then under Assumptions E, K, M, V and X,

i. Let $\chi(\kappa)$ be a Chi-square variable with κ degree of freedom. Then, for any $\kappa = \kappa_T$ in \mathcal{K}_T ,

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\frac{\varepsilon' \Omega^{-1} \Psi_\kappa (T\Sigma_\kappa)^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon - \kappa}{\sqrt{2\kappa}} \geq z \right) - \mathbb{P} \left(\frac{\chi(\kappa) - \kappa}{\sqrt{2\kappa}} \geq z \right) \right| = o(1)$$

and

$$\sup_{z \in \mathbb{R}} \left| \mathbb{P} \left(\frac{\widehat{R}_{\kappa_{\min}} - \kappa_{\min}}{\sqrt{2\kappa_{\min}}} \geq z \right) - \mathbb{P} \left(\frac{\chi(\kappa_{\min}) - \kappa_{\min}}{\sqrt{2\kappa_{\min}}} \geq z \right) \right| = o(1).$$

ii. Assume that (3.4) holds, i.e. that for some $\epsilon > 0$, $\gamma_T \geq (1 + \epsilon)\sqrt{2 \ln \text{Card} \mathcal{K}_T}$. Then

$$\mathbb{P} \left(\max_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} \frac{\widehat{R}_\kappa - \widehat{R}_{\kappa_{\min}} - (\kappa - \kappa_{\min})}{\sqrt{2(\kappa - \kappa_{\min})}} \geq \gamma_T \right) = o(1).$$

Proof of Proposition 2: see Appendix A.

Proof of Theorem 1. (3.2) and Proposition 2-ii yield

$$\mathbb{P} \left(\widehat{R}^\gamma \neq \widehat{R}_{\kappa_{\min}} \right) \leq \mathbb{P} \left(\widehat{\kappa}^\gamma \neq \kappa_{\min} \right) = \mathbb{P} \left(\max_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} \frac{\widehat{R}_\kappa - \widehat{R}_{\kappa_{\min}} - (\kappa - \kappa_{\min})}{\sqrt{2(\kappa - \kappa_{\min})}} \geq \gamma_T \right) = o(1).$$

Then the definition of z_α in (2.6) and Proposition 2-i yield

$$\mathbb{P} \left(\frac{\widehat{R}^\gamma - \kappa_{\min}}{\sqrt{2\kappa_{\min}}} \geq z_\alpha \right) = \mathbb{P} \left(\frac{\widehat{R}_{\kappa_{\min}} - \kappa_{\min}}{\sqrt{2\kappa_{\min}}} \geq z_\alpha \right) + o(1) = \mathbb{P} \left(\frac{\chi(\kappa_{\min}) - \kappa_{\min}}{\sqrt{2\kappa_{\min}}} \geq z_\alpha \right) + o(1) \rightarrow \alpha. \square$$

6.3 Proof of Theorems 2 and 3

The next Lemma is crucial for the consistency properties of the test and is used for the item \widehat{R}_{1K} in (2.3).

Lemma 1 Consider a departure from the null such that $\sup_{x \in \Lambda} |\Delta_T(x)| = O\left[\mathbb{E}^{1/2}(\Delta_T(X_t)/\sigma(X_t))^2\right]$. Assume Assumptions E, V, X hold and that $\kappa = \kappa_T$ diverges with $\kappa = o(T^{1/3}/\ln^2 T)$.

Then there exists a constant $C_5 > 0$, depending upon s, L and Λ , such that for any $\kappa \in \mathcal{K}_T$, any $\Delta(\cdot)$ from Λ to \mathbb{R} in $\mathcal{C}(L, s)$, we have

$$\left[\Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa}\right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} \Delta_T\right]^{1/2} \geq T^{1/2} \left[\mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)}\right)^2 - C_5 \kappa^{-s/d}\right] (1 + o_{\mathbb{P}}(1)), \quad (6.3)$$

$$\left|\Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa}\right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} \varepsilon\right| = T^{1/2} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)}\right)^2 + \kappa^{-s/d}\right]. \quad (6.4)$$

Proof of Lemma 1: see Appendix A.

Proof of Theorem 2. Let $s \leq d(2/C_1 - 1)$ and L be some unknown smoothness indexes. Let K_* be as in (3.6), so that K_* corresponds to a κ_* in the new indexation. Observe that this κ_* is such that

$$T \kappa_*^{-\frac{2s}{d}} \asymp T \rho_T^2 = \left(\sqrt{\ln \ln T}\right)^{\frac{4s}{4s+d}} T^{\frac{d}{4s+d}} \asymp \gamma_T \kappa_*^{1/2} \asymp \gamma_T \sqrt{2(\kappa_* - \kappa_{\min})}, \quad (6.5)$$

since the exact order of γ_T is $\ln^{1/2} \ln T$, $s > 0$ and κ_{\min} is smaller than a power of $\ln T$.

Consider now a sequence of alternatives $\mu_T(\cdot)$ in $H_1(C_3 \rho_T)$ with $C_3 \rho_T > 2C_5 \kappa_*^{-s/d}$, where C_5 is from Lemma 1. This gives that $\mathbb{E}^{1/2}(\Delta_T(X_t)/\sigma(X_t))^2 - C_5 \kappa_*^{-s/d} \geq \frac{1}{2} \mathbb{E}^{1/2}(\Delta_T(X_t)/\sigma(X_t))^2$ and that $T \mathbb{E}(\Delta_T(X_t)/\sigma(X_t))^2$ diverges. Hence Lemma 1 gives

$$\begin{aligned} \Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Delta_T &\geq \left(\frac{1}{4} + o_{\mathbb{P}}(1)\right) T \mathbb{E} \left(\frac{\Delta_T(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)}\right)^2, \\ \Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \varepsilon &= O_{\mathbb{P}} \left[T^{1/2} \mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)}\right)^2\right] \\ &= o_{\mathbb{P}}(1) \Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Delta_T. \end{aligned}$$

Observe also that Proposition 2-(i) shows that

$$\begin{aligned} \varepsilon' \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \varepsilon - \kappa_T &= O_{\mathbb{P}} \left(\kappa_*^{1/2}\right) = o_{\mathbb{P}}(1) \left(\gamma_T \sqrt{2(\kappa_* - \kappa_{\min})}\right) \\ &= o_{\mathbb{P}}(1) \Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Delta_T. \end{aligned}$$

Hence, (6.5), applying Proposition 1 for $\mathcal{K}_T = \{\kappa_*\}$ (so that $\kappa_{\max} = \kappa_{\min} = \kappa_T$) and substituting yield

$$\begin{aligned} \widehat{R}_{\kappa_*} - \kappa_* - \gamma_T \sqrt{2(\kappa_T - \kappa_{\min})} - z_{\alpha} \sqrt{2\kappa_{\min}} \\ &= \left[\Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa_*} \left(\Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Psi_{\kappa_*}\right)^{-1} \Psi'_{\kappa_*} \widehat{\Omega}^{-1} \Delta_T - \gamma_T \sqrt{2(\kappa_T - \kappa_{\min})}\right] (1 + o_{\mathbb{P}}(1)) \\ &\geq T \left[\frac{1}{4} \mathbb{E} \left(\frac{\Delta_T(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)}\right)^2 - C \rho_T^2\right] (1 + o_{\mathbb{P}}(1)) \geq T \rho_T^2 \left(\frac{C_3^2}{4} - C\right) (1 + o_{\mathbb{P}}(1)) \xrightarrow{\mathbb{P}} +\infty \end{aligned}$$

provided C_3 is large enough. The lower power bound (2.8) then shows that Theorem 2 is proven. \square

Proof of Theorem 3. Since the proof of Theorem 3 is similar to the proof of Theorem 2 up to the fact that detection is achieved through κ_{\min} , we just give the main steps. (2.7) yields that $\widehat{R}^\gamma - \kappa_{\min} \geq \widehat{R}_{\kappa_{\min}} - \kappa_{\min}$, so that it is sufficient to show that $\widehat{R}_{\kappa_{\min}} - \kappa_{\min} - \sqrt{2\kappa_{\min}}$ diverges to $+\infty$ in probability. Building on Propositions 1, 2-(i) and Lemma 1 as for Theorem 2 now gives, since $\kappa_{\min} \asymp K_{\min}^d \rightarrow \infty$,

$$\begin{aligned} \widehat{R}_{\kappa_{\min}} - \kappa_{\min} - \sqrt{2\kappa_{\min}} &\geq r_T^2 \Delta'_{0T} \widehat{\Omega}^{-1} \Psi_{\kappa_T} \left(\Psi'_{\kappa_T} \widehat{\Omega}^{-1} \Psi_{\kappa_T} \right)^{-1} \Psi'_{\kappa_T} \widehat{\Omega}^{-1} \Delta_{0T} (1 + o_{\mathbb{P}}(1)) - O_{\mathbb{P}} \left(\kappa_{\min}^{1/2} \right) \\ &\geq Tr_T^2 \left[\mathbb{E} \left(\frac{\Delta_{0T}(X_t) \mathbb{I}(X_t \in \Lambda)}{\sigma(X_t)} \right)^2 - C_5 \kappa_{\min}^{-s/d} \right] (1 + o_{\mathbb{P}}(1)) - O_{\mathbb{P}} \left(\kappa_{\min}^{1/2} \right) = Tr_T^2 - O_{\mathbb{P}} \left(K_{\min}^{d/2} \right) \xrightarrow{\mathbb{P}} +\infty \end{aligned}$$

provided Tr_T^2 diverges with $\lim_{T \rightarrow \infty} K_{\min}^{d/2} / (Tr_T^2) = 0$ as assumed in Theorem 3. \square

Appendix A: Proofs of Propositions 1, 2 and Lemma 1

A.1 Preliminary lemmas

We begin with the estimation errors $\widehat{\Sigma}_k - \Sigma_k$, see (6.2), and preliminary bounds. Define

$$\widetilde{\psi}_k(\cdot) = \psi_k(\cdot) - \Psi_{\kappa_{\min}}(\cdot) \Sigma_{\kappa_{\min}}^{-1} \mathbb{E} \left[\frac{\psi_k(X_t) \Psi'_{\kappa_{\min}}(X_t)}{\sigma^2(X_t)} \right], \quad \widetilde{\Sigma}_\kappa = \left[\mathbb{E} \left(\frac{\widetilde{\psi}_k(X_t) \widetilde{\psi}_\ell(X_t)}{\sigma^2(X_t)} \right) \right]_{\kappa_{\min} < k, \ell \leq \kappa}, \quad (\text{A.1})$$

which are used to study the difference $\widehat{R}_\kappa - \widehat{R}_{\kappa_{\min}}$ in the proof of Proposition 2-ii. The next lemmas hold for general orthonormal systems $\{\psi_k(\cdot)\}_{k \in \mathbb{N}^*}$ of $L^2(\Lambda, dx)$ with $\sup_{k \in \mathbb{N}^*} \sup_{x \in \Lambda} |\psi_k(x)| < \infty$. Recall that v_T is such that $\sup_{x \in \Lambda} |\widehat{\sigma}(x) - \sigma(x)| = O_{\mathbb{P}}(v_T)$ with $v_T = o(\kappa_{\max}^{-3/2} / \ln T)$, see Assumption V.

Lemma A.1 *Let $\Sigma_\kappa, \widehat{\Sigma}_\kappa$ be as in (6.2), and $\{\widetilde{\psi}_k(\cdot)\}_{k > \kappa_{\min}}, \widetilde{\Sigma}_\kappa$ as in (A.1). Then, under Assumptions E, V and X,*

- i. $\sup_{\kappa \in \mathbb{N}^*} \max(\|\Sigma_\kappa^{-1}\|, \|\Sigma_\kappa\|) < \infty$, $\sup_{k > \kappa_{\min}} \sup_{x \in \Lambda} |\widetilde{\psi}_k(x)| < C \kappa_{\min}^{1/2}$ and $\sup_{\kappa \in \mathbb{N}^*} \max(\|\widetilde{\Sigma}_\kappa^{-1}\|, \|\widetilde{\Sigma}_\kappa\|) < \infty$.
- ii. If $\kappa_{\max} = o(T^{1/2})$, the matrices $\widehat{\Sigma}_\kappa$, $1 \leq \kappa \leq \kappa_{\max}$, have an inverse with a probability tending to 1 and

$$\max_{\kappa \in \mathcal{K}_T} \max \left[\|\widehat{\Sigma}_\kappa - \Sigma_\kappa\|, \|\widehat{\Sigma}_\kappa^{-1} - \Sigma_\kappa^{-1}\| \right] = O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^2}{T} \right)^{1/2} + \kappa_{\max} v_T \right] = o_{\mathbb{P}}(1).$$

- iii. If $\kappa_{\max} = o(T^{1/2})$, $\max_{1 \leq \kappa \leq \kappa_{\max}} \max(\|\widehat{\Sigma}_\kappa\|, \|\widehat{\Sigma}_\kappa^{-1}\|) = O_{\mathbb{P}}(1)$.

Lemma A.2 *Let $m_T(\cdot)$ and $\mu_T(\cdot)$ from \mathbb{R}^d to \mathbb{R} be some functions with support Λ . Then, under Assumptions E, V, X and if $\text{Card} \mathcal{K}_T = O(\ln T)$, $\kappa_{\max} = o(T^{1/3} / \ln^{2/3} T)$*

$$\max_{\kappa \in \mathcal{K}_T} \left| \mu_T \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} m_T \right| \leq \left\| \widehat{\Omega}^{-1/2} \mu_T \right\| \left\| \widehat{\Omega}^{-1/2} m_T \right\|, \quad (\text{A.2})$$

$$\max_{\kappa \in \mathcal{K}_T} \frac{\left\| \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon \right\|}{\sqrt{T \kappa}} = \text{Card}^{1/2} \mathcal{K}_T O_{\mathbb{P}}(1), \quad (\text{A.3})$$

$$\begin{aligned}
& \max_{\kappa \in \mathcal{K}_T} \frac{\left| \varepsilon' \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon - \varepsilon' \Omega^{-1} \Psi_\kappa \left(T \Sigma_\kappa \right)^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon \right|}{\kappa^{1/2}} \\
&= \text{Card} \mathcal{K}_T O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^3}{T} \right)^{1/2} + \kappa_{\max}^{3/2} v_T \right] = o_{\mathbb{P}}(1), \tag{A.4}
\end{aligned}$$

$$\begin{aligned}
& \max_{\kappa \in \mathcal{K}_T} \frac{\left| m'_T \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon \right|}{\kappa^{1/2}} \\
&= \frac{T^{1/2}}{\kappa_{\min}^{1/2}} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{m_T(X_t)}{\sigma(X_t)} \right)^2 + \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| + v_T \sup_{x \in \Lambda} |m_T(x)| \right]. \tag{A.5}
\end{aligned}$$

The functions $m_T(\cdot)$ and $\mu_T(\cdot)$ may depend upon $(X_1, \varepsilon_1), \dots, (X_T, \varepsilon_T)$ in (A.2) but not in (A.5).

Proof of Lemmas A.1 and A.2: see Appendix B.

The next lemma is used for Proposition 2. It is stated for general maps $\varphi_k(\cdot)$ from \mathbb{R}^d to \mathbb{R} , $k \geq 1$. Consider the row vector $\Phi_\kappa(X_t) = [\varphi_1(X_t), \dots, \varphi_\kappa(X_t)]$ and the $\kappa \times T$ matrix $\Phi_\kappa = [\Phi_\kappa(X_1)' \dots, \Phi_\kappa(X_T)']'$. Define

$$V_\kappa = \mathbb{E} \left[\frac{\Phi'_\kappa(X_t) \Phi_\kappa(X_t)}{\sigma(X_t)} \right].$$

We assume

Assumption B The matrices V_κ have an inverse with $\sup_{\kappa \in \mathbb{N}^*} \|V_\kappa^{-1}\| < \infty$ and the functions $\varphi_k(\cdot)$ are such that $\max(\sup_{1 \leq k \leq \kappa} \sup_{x \in \mathbb{R}^d} |\varphi_k(x)|, 1) = \varphi_\infty < \infty$.

Define

$$S_T = S_{\kappa T} = V_\kappa^{-1/2} \sum_{t=1}^T \frac{\Phi'_\kappa(X_t)}{\sigma(X_t)} \frac{\varepsilon_t}{\sigma(X_t)}, \quad Q_T = Q_{\kappa T} = \frac{T^{-1} S'_T S_T - \kappa}{\sqrt{2\kappa}} = \frac{T^{-1} \|S_T\|^2 - \kappa}{\sqrt{2\kappa}}.$$

We now study the tail probability of Q_T .

Lemma A.3 Let $Q_T = Q_{\kappa T}$ be as above. Then, under Assumptions E, X-i, B, and $\kappa = \kappa_T = o(T^{\frac{3}{4} \frac{1+a}{5+3a}})$,

i. Let $\chi(\kappa)$ be a Chi-square variable with κ degree of freedom. Then

$$\lim_{T \rightarrow \infty} \sup_{\gamma \in \mathbb{R}} \left| \mathbb{P}(Q_T \geq \gamma) - \mathbb{P}\left(\frac{\chi(\kappa) - \kappa}{\sqrt{2\kappa}} \geq \gamma\right) \right| = 0.$$

ii. Consider $\epsilon > 0$. Then there exists a constant C_ϵ , which does not depend upon κ and γ , such that for any $\gamma > \epsilon$ and κ ,

$$\mathbb{P}(Q_T \geq \gamma) \leq \frac{1}{\sqrt{2\pi}(\gamma - \epsilon)} \exp\left(-\frac{(\gamma - \epsilon)^2}{2}\right) + C_\epsilon \left[\varphi_\infty^6 \kappa^2 T^{-\frac{3}{2} \frac{1+a}{5+3a}} + \frac{1}{\sqrt{\kappa}} \right].$$

Proof of Lemma A.3: see Appendix B.

A.2 Proof of Propositions 1 and 2

Proof of Proposition 1. For brevity of notations, the proof is made for $p = \dim\theta = 1$. Define

$$e(\theta) = [e_1(\theta), \dots, e_T(\theta)]' \text{ where } e_t = m(X_t; \theta_T) - m(X_t; \hat{\theta}_T) \text{ so that } \hat{U} = U + e(\theta).$$

This gives

$$\begin{aligned} \hat{R}_\kappa &= U' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} U + 2A_\kappa + B_\kappa \text{ with } A_\kappa = U' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} e(\theta) \\ &\text{and } B_\kappa = e(\theta)' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} e(\theta). \end{aligned}$$

Under Assumption M, $\max_{1 \leq t \leq T} |e_t(\theta)| = O_{\mathbb{P}}(T^{-1/2})$ which gives $\|e(\theta)\| = O_{\mathbb{P}}(1)$ and $\max_{\kappa \in \mathcal{K}_T} |B_\kappa| = O_{\mathbb{P}}(1)$, so that $\max_{1 \leq \kappa \leq \kappa_{\max}} \kappa^{-1/2} |B_\kappa| = O_{\mathbb{P}}(\kappa_{\min}^{-1/2})$. Consider now A_κ . Under Assumption M, the Taylor formula gives

$$e_t(\theta) = \left(\hat{\theta}_T - \theta_T \right) \frac{\partial m(X_t; \theta_T)}{\partial \theta} + \frac{1}{2} \left(\hat{\theta}_T - \theta_T \right)^2 \frac{\partial^2 m(X_t; \theta_T^*)}{\partial^2 \theta} \text{ so that } e(\theta) = \left(\hat{\theta}_T - \theta_T \right) m_1 + \frac{1}{2} \left(\hat{\theta}_T - \theta_T \right)^2 m_2,$$

with a θ_T^* between θ_T and $\hat{\theta}_T$ and where m_1 and m_2 are \mathbb{R}^T columns vector with bounded entries given by the first and second-order derivatives. Since $U = \Delta_T + \varepsilon$, this gives

$$\begin{aligned} A_\kappa &= A_{1\kappa} + A_{2\kappa} + \frac{1}{2} A_{3\kappa} \text{ with } A_{1\kappa} = e'(\theta) \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \Delta_T, \\ A_{2\kappa} &= \left(\hat{\theta}_T - \theta_T \right) m_1' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \varepsilon, \quad A_{3\kappa} = \left(\hat{\theta}_T - \theta_T \right)^2 m_2' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \varepsilon. \end{aligned}$$

The Cauchy-Schwarz inequality gives $|A_{1\kappa}| \leq \|e(\theta)\| \|\Delta_T\|$ with $\|e(\theta)\| = O_{\mathbb{P}}(1)$, so that

$$\max_{\kappa \in \mathcal{K}_T} \frac{|A_{1\kappa}|}{\kappa^{1/2}} = O_{\mathbb{P}} \left(\frac{\|\Delta_T\|}{\kappa_{\min}^{1/2}} \right) = O_{\mathbb{P}} \left(\frac{T^{1/2}}{\kappa_{\min}^{1/2}} \mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)} \right)^2 \right),$$

since $\|\Delta_T\|^2 = O_{\mathbb{P}}(T) \mathbb{E}(\Delta_T(X_t)/\sigma(X_t))^2$ by the Markov inequality and Assumption E. Since $T^{1/2}(\hat{\theta}_T - \theta_T) = O_{\mathbb{P}}(1)$ and under Assumption M, applying (A.5) for $A_{2\kappa}$ and the Cauchy-Schwarz inequality for $A_{3\kappa}$ give

$$\begin{aligned} \max_{\kappa \in \mathcal{K}_T} \frac{|A_{2\kappa}|}{\kappa^{1/2}} &= O_{\mathbb{P}} \left(\kappa_{\min}^{-1/2} \right), \\ \max_{\kappa \in \mathcal{K}_T} \frac{|A_{3\kappa}|}{\kappa^{1/2}} &= O_{\mathbb{P}} \left(\frac{1}{T \kappa_{\min}^{1/2}} \right) \|m_2\| \|\varepsilon\| = O_{\mathbb{P}} \left(\kappa_{\min}^{-1/2} \right). \end{aligned}$$

Substituting in the expression of A_κ and \hat{R}_κ give

$$\max_{\kappa \in \mathcal{K}_T} \frac{\left| \hat{R}_\kappa - U' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} U \right|}{\kappa^{1/2}} = \frac{1}{\kappa_{\min}^{1/2}} O_{\mathbb{P}} \left[1 + T^{1/2} \mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)} \right)^2 \right]. \quad (\text{A.6})$$

But

$$\begin{aligned} U' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} U &= e' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \varepsilon \\ &\quad + 2 \Delta_T' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \varepsilon + \Delta_T' \hat{\Omega}^{-1} \Psi_\kappa \left(T \hat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \hat{\Omega}^{-1} \Delta_T \end{aligned}$$

so that substituting (A.4) in the equation above and (A.6) give the desired result. \square

Proof of Proposition 2. Define

$$R_\kappa^0 = \varepsilon' \Omega^{-1} \Psi_\kappa (T \Sigma_\kappa)^{-1} \Psi_\kappa' \Omega^{-1} \varepsilon \text{ and } Q_\kappa^0 = Q_{\kappa T}^0 = \frac{R_\kappa^0 - \kappa}{\sqrt{2\kappa}}.$$

Under the null, Proposition 1 yields

$$\max_{\kappa \in \mathcal{K}_T} \left| \frac{\widehat{R}_\kappa - R_\kappa^0}{\kappa^{1/2}} \right| = o_{\mathbb{P}}(1) \text{ or, equivalently, } \max_{\kappa \in \mathcal{K}_T} \left| \frac{\widehat{R}_\kappa - \kappa}{\sqrt{2\kappa}} - Q_\kappa^0 \right| = o_{\mathbb{P}}(1). \quad (\text{A.7})$$

Hence Proposition 2-(i) follows from taking $\kappa = \kappa_{\min}$ in Lemma A.3-(i) and (A.7). Consider now Proposition 2-(ii). Let ϵ be as in (3.4), so that $\gamma_T \geq \sqrt{2 \ln \text{Card} \mathcal{K}_T} + \epsilon$ for T large enough. Therefore (A.7) yields that Proposition 2-(ii) is a consequence of

$$\mathbb{P} \left(\max_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} \frac{R_\kappa^0 - R_{\kappa_{\min}}^0 - (\kappa - \kappa_{\min})}{\sqrt{2(\kappa - \kappa_{\min})}} \geq \sqrt{2 \ln \text{Card} \mathcal{K}_T} + \epsilon \right) = o(1). \quad (\text{A.8})$$

To prove (A.8), we first rewrite $R_\kappa^0 - R_{\kappa_{\min}}^0$ as suitable quadratic form. For $k, \kappa > \kappa_{\min}$, let $\tilde{\psi}_k(\cdot)$ and $\tilde{\Sigma}_\kappa$ be as in (A.1), and consider the rows vectors $\tilde{\Psi}_{\kappa_{\min}}^\kappa(X_t) = [\tilde{\psi}_{\kappa_{\min}+1}(X_t), \dots, \tilde{\psi}_\kappa(X_t)]$,

$$\tilde{\Psi}_\kappa(X_t) = [\Psi_{\kappa_{\min}}(X_t), \tilde{\Psi}_{\kappa_{\min}}^\kappa(X_t)] = \Psi_\kappa(X_t) \beta_\kappa \text{ so that } \tilde{\Psi}_\kappa = \Psi_\kappa \beta_\kappa,$$

for some regular $\kappa \times \kappa$ matrix β_κ . Elementary algebra gives

$$R_\kappa^0 = T^{-1} \varepsilon' \Omega^{-1} \tilde{\Psi}_\kappa \begin{bmatrix} \Sigma_{\kappa_{\min}}^{-1} & 0 \\ 0 & \tilde{\Sigma}_\kappa^{-1} \end{bmatrix} \tilde{\Psi}_\kappa' \Omega^{-1} \varepsilon \text{ and } R_{\kappa_{\min}}^0 = T^{-1} \varepsilon' \Omega^{-1} \tilde{\Psi}_\kappa \begin{bmatrix} \Sigma_{\kappa_{\min}}^{-1} & 0 \\ 0 & 0 \end{bmatrix} \tilde{\Psi}_\kappa' \Omega^{-1} \varepsilon.$$

Hence

$$\begin{aligned} R_\kappa^0 - R_{\kappa_{\min}}^0 &= T^{-1} \varepsilon' \Omega^{-1} \tilde{\Psi}_\kappa \begin{bmatrix} 0 & 0 \\ 0 & \tilde{\Sigma}_\kappa^{-1} \end{bmatrix} \tilde{\Psi}_\kappa' \Omega^{-1} \varepsilon = T^{-1} \varepsilon' \Omega^{-1} \tilde{\Psi}_{\kappa_{\min}}^\kappa \tilde{\Sigma}_\kappa^{-1} \left[\tilde{\Psi}_{\kappa_{\min}}^\kappa \right]' \Omega^{-1} \varepsilon \\ &= \sqrt{2(\kappa - \kappa_{\min})} \tilde{Q}_\kappa + \kappa - \kappa_{\min}. \end{aligned}$$

We now verify that the quadratic form \tilde{Q}_κ obeys the conditions of Lemma A.3. Lemma A.1-(i) yields that $\sup_\kappa \sup_{x \in \mathbb{R}^d} |\tilde{\psi}_\kappa(x)| \leq C \kappa_{\min}^{1/2}$, so that Assumption B holds taking $\varphi_\infty = O(\kappa_{\min}^{1/2}) = O(\ln^{C_2 d/2} T)$. Recall that $\kappa - \kappa_{\min} \asymp 2^{jd} - 2^{J_{\min} d}$ by the definition (3.1) of \mathcal{K}_T . Hence Lemma A.3-(ii) yields, for (A.8),

$$\begin{aligned} \mathbb{P} \left(\max_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} \tilde{Q}_\kappa \geq \sqrt{2 \ln \text{Card} \mathcal{K}_T} + \epsilon \right) &\leq \sum_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} \mathbb{P} \left(\tilde{Q}_\kappa \geq \sqrt{2 \ln \text{Card} \mathcal{K}_T} + \epsilon \right) \\ &\leq \text{Card} \mathcal{K}_T \frac{\exp(-\ln \text{Card} \mathcal{K}_T)}{2\pi \sqrt{2 \ln \text{Card} \mathcal{K}_T}} + C \varphi_\infty^6 \sum_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} (\kappa - \kappa_{\min})^2 T^{-\frac{3}{2} \frac{1+a}{5+3a}} + C \sum_{\kappa \in \mathcal{K}_T \setminus \{\kappa_{\min}\}} (\kappa - \kappa_{\min})^{-1/2} \\ &= o(1) + C \varphi_\infty^6 \text{Card} \mathcal{K}_T \kappa_{\max} T^{-\frac{3}{2} \frac{1+a}{5+3a}} + C 2^{-d J_{\min}/2} \sum_{j=1}^{+\infty} 2^{-jd/2} = o(1). \quad \square \end{aligned}$$

A.3 Proof of Lemma 1

In this proof, we apply lemmas A.1 and A.2 for $\mathcal{K}_T = \{\kappa\}$, which is such that $\kappa = \kappa_{\min} = \kappa_{\max} = o(T^{1/3}/\ln^2 T)$. The Jackson Theorem (see Timan (1994), eq. (8) p. 278) yields that there is a trigonometric polynomial function $\Pi(\cdot) = \Pi_{\Delta_T, \kappa}(\cdot)$ with degree $\asymp \kappa^{1/d}$ such that

$$\Pi(x) = \sum_{k=1}^{\kappa} \beta_k \psi_k(x) \mathbb{I}(x \in \Lambda) \text{ such that } \sup_{x \in \Lambda} |\Delta_T(x) - \Pi(x)| \leq C\kappa^{-s/d}. \quad (\text{A.9})$$

Since $\widehat{\sigma}(\cdot)$ is bounded away from 0 over Λ in probability, (A.9) implies that

$$\max_{1 \leq t \leq T} \left| \frac{\Delta_T(X_t) - \Pi(X_t)}{\widehat{\sigma}(X_t)} \right| = O_{\mathbb{P}}(\kappa^{-s/d}).$$

Note that $\left| m' \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right) \Psi'_{\kappa} \widehat{\Omega}^{-1} m \right| \leq \|m\| \leq T^{1/2} \sup_{x \in \mathbb{R}^d} |m(x)|$. Let $\Pi = [\Pi(X_1), \dots, \Pi(X_T)]'$, which is such that $\Pi' \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right) \Psi'_{\kappa} \widehat{\Omega}^{-1} \Pi = \Pi' \widehat{\Omega}^{-1} \Pi = \left\| \widehat{\Omega}^{-1/2} \Pi \right\|^2$ since $\widehat{\Omega}^{-1/2} \Pi$ is in the space spanned by the columns of $\widehat{\Omega}^{-1/2} \Psi_{\kappa}$. Hence the triangular inequality and (A.9) give

$$\begin{aligned} \left[\Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} \Delta_T \right]^{1/2} &\geq \left[\Pi' \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} \Pi \right]^{1/2} \\ &\quad - \left[(\Pi - \Delta_T)' \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} (\Pi - \Delta_T) \right]^{1/2} \\ &\geq \left\| \widehat{\Omega}^{-1/2} \Pi \right\| - CT^{1/2} \kappa^{-s/d}. \end{aligned}$$

In the expression (A.9) of $\Pi(\cdot)$, write $\beta = [\beta_1, \dots, \beta_{\kappa}]'$, so that the definitions of $\widehat{\Sigma}_{\kappa}$, Σ_{κ} in (6.2) and Lemma A.1-(ii) give

$$\begin{aligned} \left\| \widehat{\Omega}^{-1/2} \Pi \right\| &= \left(\sum_{t=1}^T \frac{\Pi^2(X_t)}{\widehat{\sigma}^2(X_t)} \right)^{1/2} = (T \beta' \widehat{\Sigma}_{\kappa} \beta)^{1/2} = T (\beta' \Sigma_{\kappa} \beta)^{1/2} \left(1 + O_{\mathbb{P}} \left[\left(\frac{\kappa^2}{T} \right)^{1/2} + \kappa v_T \right] \right) \\ &= T^{1/2} \mathbb{E}^{1/2} \left(\frac{\Pi(X_t)}{\sigma(X_t)} \right)^2 (1 + o_{\mathbb{P}}(1)) \geq T^{1/2} \left[\mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)} \right)^2 - C\kappa^{-s/d} \right]. \end{aligned}$$

Substituting shows that (6.3) is proven. (6.4) follows from (A.5) and Assumption V, which gives

$$\left| \Delta'_T \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi'_{\kappa} \widehat{\Omega}^{-1} \Psi_{\kappa} \right)^{-1} \Psi'_{\kappa} \widehat{\Omega}^{-1} \varepsilon \right| = T^{1/2} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{\Delta_T(X_t)}{\sigma(X_t)} \right)^2 + \kappa^{-s/d} \right]. \quad \square$$

Appendix B: Proof of Lemmas A.1-A.3

B.1 Proof of Lemma A.1

We begin with Lemma A.1-(i), $\sup_{\kappa \in \mathbb{N}^*} \max(\|\Sigma_{\kappa}^{-1}\|, \|\Sigma_{\kappa}\|) < \infty$. Since $u' \Sigma_{\kappa} u = \mathbb{E} \left(\sum_{k=1}^{\kappa} u_k \psi_k(X_t) / \sigma(X_t) \right)^2$, $\|\Sigma_{\kappa}\|$ is the largest eigenvalue of the symmetric Σ_{κ} and $\|\Sigma_{\kappa}^{-1}\|$ is the inverse of the smallest eigenvalue of Σ_{κ} . Hence

$$\|\Sigma_{\kappa}\| = \sup_{\|u\|=1} \mathbb{E} \left(\sum_{k=1}^{\kappa} u_k \frac{\psi_k(X_t)}{\sigma(X_t)} \right)^2, \quad \frac{1}{\|\Sigma_{\kappa}^{-1}\|} = \inf_{\|u\|=1} \mathbb{E} \left(\sum_{k=1}^{\kappa} u_k \frac{\psi_k(X_t)}{\sigma(X_t)} \right)^2,$$

Since $f(\cdot)$ and $\sigma(\cdot)$ are bounded away from 0 and infinity over Λ by Assumptions E and X-ii, and because $\{\psi_k(\cdot)\}_{k \in \mathbb{N}^*}$ is an orthonormal system of $L^2(\Lambda, dx)$, we have uniformly in κ

$$\mathbb{E} \left(\sum_{k=1}^{\kappa} u_k \frac{\psi_k(X_t)}{\sigma(X_t)} \right)^2 = \int_{\Lambda} \left(\sum_{k=1}^{\kappa} u_k \psi_k(x) \right)^2 \frac{f(x)}{\sigma^2(x)} dx \asymp \int_{\Lambda} \left(\sum_{k=1}^{\kappa} u_k \psi_k(x) \right)^2 dx = \|u\|^2.$$

This gives $\sup_{\kappa \in \mathbb{N}^*} \max(\|\Sigma_{\kappa}^{-1}\|, \|\Sigma_{\kappa}\|) < \infty$, and we now prove that $\sup_{\kappa \in \mathbb{N}^*} \max(\|\tilde{\Sigma}_{\kappa}^{-1}\|, \|\tilde{\Sigma}_{\kappa}\|) < \infty$. Let $\Psi_{\kappa_{\min}}^{\kappa}(X_t) = [\psi_{\kappa_{\min}+1}(X_t), \dots, \psi_{\kappa}(X_t)]$ and note that

$$\tilde{\Sigma}_{\kappa} = \left[\mathbb{E} \left(\frac{\psi_k(X_t) \psi_{\ell}(X_t)}{\sigma^2(X_t)} \right) \right]_{\kappa_{\min} < k, \ell \leq \kappa} - \mathbb{E} \left(\frac{\Psi_{\kappa_{\min}}^{\kappa}(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right) \Sigma_{\kappa_{\min}}^{-1} \mathbb{E} \left(\frac{\Psi_{\kappa_{\min}}^{\kappa}(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right).$$

It then follows that $\tilde{\Sigma}_{\kappa} \preceq [\mathbb{E} \psi_k(X_t) \psi_{\ell}(X_t) / \sigma^2(X_t)]_{\kappa_{\min} < k, \ell \leq \kappa}$ where $A \preceq B$ means that $A - B$ is a symmetric non negative matrix. This gives that $\|\tilde{\Sigma}_{\kappa}\| \leq \|\Sigma_{\kappa}\|$ since the upper bound is a diagonal block submatrix of Σ_{κ} . Observe that $\tilde{\Sigma}_{\kappa}^{-1}$ is also a diagonal block of Σ_{κ}^{-1} by the partitioned inverse formula, so that $\|\tilde{\Sigma}_{\kappa}^{-1}\| \leq \|\Sigma_{\kappa}^{-1}\|$. This gives $\sup_{\kappa \in \mathbb{N}^*} \max(\|\tilde{\Sigma}_{\kappa}^{-1}\|, \|\tilde{\Sigma}_{\kappa}\|) < \infty$. To show that $\sup_{k > \kappa_{\min}} \sup_{x \in \Lambda} |\tilde{\psi}_k(x)| < \infty$, note that $\Psi_{\kappa_{\min}}^{\kappa}(\cdot) \Sigma_{\kappa_{\min}}^{-1} \mathbb{E}[\psi_k(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t) / \sigma^2(X_t)]$ is the $L_2(\Lambda, f(x) dx / \sigma^2(x))$ -orthogonal projection of $\psi_k(\cdot)$ on $\psi_1(\cdot), \dots, \psi_{\kappa_{\min}}(\cdot)$. The Pythagore inequality gives, uniformly in $k \geq 1$,

$$\mathbb{E} \left(\frac{\Psi_{\kappa_{\min}}^{\kappa}(X_t) \Sigma_{\kappa_{\min}}^{-1} \mathbb{E} \left[\frac{\psi_k(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right]}{\sigma(X_t)} \right)^2 \leq \mathbb{E} \left[\frac{\psi_k^2(X_t)}{\sigma^2(X_t)} \right] \leq C.$$

Therefore, the Cauchy-Schwartz inequality gives for all x and $\kappa \geq 1$,

$$\begin{aligned} |\tilde{\psi}_k(x)| &\leq \sup_{\kappa \geq 1} \sup_{x \in \Lambda} |\psi_{\kappa}(x)| + \sup_{x \in \Lambda} \|\Sigma_{\kappa_{\min}}^{-1/2} \Psi_{\kappa_{\min}}^{\prime}(x)\| \left\| \Sigma_{\kappa_{\min}}^{-1/2} \mathbb{E} \left[\frac{\psi_k(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right] \right\| \\ &\leq C + C \|\Sigma_{\kappa_{\min}}^{-1/2}\| \kappa_{\min}^{1/2} \times \mathbb{E} \left[\frac{\psi_k(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right] \Sigma_{\kappa_{\min}}^{-1} \mathbb{E} \left[\frac{\psi_k(X_t) \Psi_{\kappa_{\min}}^{\prime}(X_t)}{\sigma^2(X_t)} \right] \\ &\leq C \kappa_{\min}^{1/2} \mathbb{E} \psi_k^2(X_t) = C \kappa_{\min}^{1/2}. \end{aligned}$$

Consider now Lemma A.1-(ii,iii). Define

$$\bar{\Sigma}_{\kappa} = \widehat{\Sigma}_{\kappa}(\Omega) = \left[\frac{\psi_k(X) \psi_{\ell}(X)}{\sigma^2(X)} = \bar{\Sigma}_{k\ell} \right]_{1 \leq k, \ell \leq \kappa}.$$

Assumptions E, X-i and (6.1) give

$$\mathbb{E} \bar{\Sigma}_{k\ell} = \mathbb{E} \left[\frac{\psi_k(X_t) \psi_{\ell}(X_t)}{\sigma^2(X)} \right] = \Sigma_{k\ell}, \quad \text{Var}(\bar{\Sigma}_{k\ell}) \leq \frac{C}{T} \sum_{n=0}^{\infty} \alpha(n),$$

and then, by the Cauchy-Schwartz inequality

$$\begin{aligned} \mathbb{E} \max_{1 \leq \kappa \leq \kappa_{\max}} \|\bar{\Sigma}_{\kappa} - \Sigma_{\kappa}\|^2 &\leq \mathbb{E} \|\bar{\Sigma}_{\kappa_{\max}} - \Sigma_{\kappa_{\max}}\|^2 = \mathbb{E} \sup_{\|u\|=1} \left[\sum_{k=1}^{\kappa_{\max}} \left(\sum_{\ell=1}^{\kappa_{\max}} (\bar{\Sigma}_{k\ell} - \Sigma_{k\ell}) u_{\ell} \right)^2 \right] \\ &\leq \sup_{\|u\|=1} \sum_{k=1}^{\kappa_{\max}} \sum_{\ell=1}^{\kappa_{\max}} \mathbb{E} (\bar{\Sigma}_{k\ell} - \Sigma_{k\ell})^2 \|u\|^2 = O\left(\frac{\kappa_{\max}^2}{T}\right) \end{aligned} \quad (\text{B.1})$$

and then $\max_{1 \leq \kappa \leq \kappa_{\max}} \|\bar{\Sigma}_\kappa - \Sigma_\kappa\| = O_{\mathbb{P}}\left(\frac{\kappa_{\max}^2}{T}\right)^{1/2}$, and we now bound $\max_{1 \leq \kappa \leq \kappa_{\max}} \|\hat{\Sigma}_\kappa - \bar{\Sigma}_\kappa\|$. We have, uniformly in $k \leq \kappa_{\max}$,

$$\begin{aligned} \|\hat{\Sigma}_\kappa - \bar{\Sigma}_\kappa\| &= \sup_{\|u\|=1} \left| \frac{u' \Psi'_\kappa (\hat{\Omega}^{-1} - \Omega^{-1}) \Psi_\kappa u}{T} \right| = \sup_{\|u\|=1} \left| \frac{1}{T} \sum_{t=1}^T \left(\sum_{k=1}^{\kappa} u_k \psi_k(X_t) \right)^2 \left(\frac{1}{\hat{\sigma}^2(X_t)} - \frac{1}{\sigma^2(X_t)} \right) \right| \\ &\leq O_{\mathbb{P}} \left(\max_{1 \leq t \leq T} |\hat{\sigma}(X_t) - \sigma(X_t)| \right) \frac{1}{T} \sum_{t=1}^T \sum_{k=1}^{\kappa_{\max}} \psi_k^2(X_t) = O_{\mathbb{P}}(\kappa_{\max} v_T). \end{aligned}$$

Since $\kappa_{\max} \asymp K_{\max}^d$, Assumption V and $\kappa_{\max}^2/T = o(1)$ yield

$$\max_{1 \leq \kappa \leq \kappa_{\max}} \|\hat{\Sigma}_\kappa - \Sigma_\kappa\| = O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^2}{T} \right)^{1/2} + \kappa_{\max} v_T \right] = o_{\mathbb{P}}(1).$$

Therefore the smallest eigenvalue of $\hat{\Sigma}_\kappa$ is bounded away from 0 and these matrices have an inverse for $1 \leq \kappa \leq \kappa_{\max}$ with a probability tending to 1. The order of $\max_{1 \leq \kappa \leq \kappa_{\max}} \|\hat{\Sigma}_\kappa^{-1} - \Sigma_\kappa^{-1}\|$ comes from the series expansion

$$\begin{aligned} \|\hat{\Sigma}_\kappa^{-1} - \Sigma_\kappa^{-1}\| &= \left\| \Sigma_\kappa^{-1} \left[\left(\text{Id}_\kappa + (\hat{\Sigma}_\kappa - \Sigma_\kappa) \Sigma_\kappa^{-1} \right)^{-1} - \text{Id}_\kappa \right] \right\| = \left\| \sum_{n=1}^{\infty} \Sigma_\kappa^{-1} \left((\hat{\Sigma}_\kappa - \Sigma_\kappa) \Sigma_\kappa^{-1} \right)^n \right\| \\ &\leq \sum_{n=1}^{\infty} \|\hat{\Sigma}_\kappa - \Sigma_\kappa\|^n \left(\sup_{\kappa \in \mathbb{N}^*} \|\Sigma_\kappa^{-1}\| \right)^{n+1}, \end{aligned}$$

which ends the proof of Lemma A.1-(i,iii) since $\sup_{\kappa} \|\Sigma_\kappa^{-1}\| < \infty$. \square

B.2 Proof of Lemma A.2

Let us recall some results from empirical process useful to establish some preliminary bounds. Consider the class of functions \mathcal{G}_T from Λ to \mathbb{R} with

$$\mathcal{G}_T = \left\{ g(\cdot) : \sup_{x \in \Lambda} |g(x) - \sigma(x)| \leq M_T, \sup_{x \in \Lambda} \left| \frac{\partial^\ell g(x)}{\partial \ell^1 x_1 \dots \partial \ell^d x_d} \right| \leq M_T \text{ for all } d\text{-uple with } \ell_1 + \dots + \ell_d = \ell \right\},$$

with ℓ as in Assumption V. Under Assumption V, there is a $M_T \asymp v_T$ such that

$$\liminf_{T \rightarrow \infty} \mathbb{P}(\hat{\sigma}^{-2}(\cdot) \in \mathcal{G}_T) \geq 1 - \epsilon, \text{ for any } \epsilon.$$

Then, to establish Lemma A.2, we can view $\hat{\sigma}^{-2}(\cdot)$ as a member of a \mathcal{G}_T . Consider now a sequence of functions from Λ to \mathbb{R} and define the empirical process $Z_T^k(\cdot) = \{Z_T^k(g), g \in \mathcal{G}_T\}$ as

$$Z_T^k(g) = \frac{1}{T^{1/2}} \sum_{t=1}^T (m_T(X_t) \psi_k(X_t) g(X_t) - \mathbb{E}[m_T(X_t) \psi_k(X_t) g(X_t)]) \text{ or } Z_T^k(g) = \frac{1}{T^{1/2}} \sum_{t=1}^T m_T(X_t) \psi_k(X_t) g(X_t) \varepsilon_t.$$

Modifications of bounds (8.3), (8.7) and (8.9) in Rio (2000) to account for multiplication by $m_T(\cdot)$ and $\psi_k(\cdot)$ with $\sup_{x \in \Lambda} |\psi_k(x)| = 1$ shows that

$$\sup_{k \geq 1} \mathbb{E} \left(\sup_{g \in \mathcal{G}_T} \left| Z_T^k(g) - Z_T^k(\sigma^{-2}) \right|^2 \right) \leq O(v_T^2) \sup_{x \in \Lambda} |m_T(x)|^2. \quad (\text{B.2})$$

Define

$$e_\kappa(\varepsilon) = \Psi'_\kappa \left(\widehat{\Omega}^{-1} - \Omega^{-1} \right) \varepsilon, \quad e_\kappa(m) = \Psi'_\kappa \left(\widehat{\Omega}^{-1} - \Omega^{-1} \right) m_T, \quad e_\kappa(\Sigma) = \widehat{\Sigma}_\kappa^{-1} - \Sigma_\kappa^{-1},$$

so that $\Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon = \Psi'_\kappa \Omega^{-1} \varepsilon + e_\kappa(\varepsilon)$, $\Psi'_\kappa \widehat{\Omega}^{-1} m_T = \Psi'_\kappa \Omega^{-1} m_T + e_\kappa(m_T)$ and $\widehat{\Sigma}_\kappa^{-1} = \Sigma_\kappa^{-1} + e_\kappa(\Sigma)$. The Chebychev inequality, (B.2) and Lemma A.1-ii give

$$\begin{aligned} \max_{\kappa \in \mathcal{K}_T} \frac{\|e_\kappa(\varepsilon)\|^2}{T\kappa} &= \max_{\kappa \in \mathcal{K}_T} \frac{1}{\kappa} \sum_{k=1}^{\kappa} \left(\frac{1}{T^{1/2}} \sum_{t=1}^T \psi_k(X_t) (\widehat{\sigma}^{-2}(X_t) - \sigma^{-2}(X_t)) \varepsilon_t \right)^2 \\ &= O_{\mathbb{P}}(1) \sum_{\kappa \in \mathcal{K}_T} \frac{1}{\kappa} \sum_{g(\cdot) \in \mathcal{G}_T} \max_{k=1}^{\kappa} \left(\frac{1}{T^{1/2}} \sum_{t=1}^T \psi_k(X_t) (g(X_t) - \sigma^{-2}(X_t)) \varepsilon_t \right)^2 = O_{\mathbb{P}}(v_T^2 \text{Card} \mathcal{K}_T), \end{aligned} \quad (\text{B.3})$$

$$\begin{aligned} \max_{\kappa \in \mathcal{K}_T} \frac{\|e_\kappa(m)\|^2}{T^2\kappa} &= O_{\mathbb{P}}(1) \max_{\kappa \in \mathcal{K}_T} \frac{1}{\kappa} \sum_{k=1}^{\kappa} \max_{g(\cdot) \in \mathcal{G}_T} \left(\frac{1}{T} \sum_{t=1}^T m_T(X_t) \psi_k(X_t) (g(X_t) - \sigma^{-2}(X_t)) \right)^2 \\ &\leq O_{\mathbb{P}}(1) \sum_{\kappa \in \mathcal{K}_T} \frac{2}{\kappa} \sum_{g \in \mathcal{G}_T} \max_{k=1}^{\kappa} \left[\left(\frac{1}{T} \sum_{t=1}^T \mathbb{E} [m_T(X_t) \psi_k(X_t) (g(X_t) - \sigma^{-2}(X_t))] \right)^2 + \frac{(Z_n^k(g) - Z_n^k(\sigma^{-2}))^2}{T^{1/2}} \right] \\ &= O_{\mathbb{P}}(v_T^2 \text{Card} \mathcal{K}_T) \sup_{x \in \Lambda} |m_T(x)|^2, \end{aligned} \quad (\text{B.4})$$

$$\max_{\kappa \in \mathcal{K}_T} \|e_\kappa(\Sigma)\| = O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^2}{T} \right)^{1/2} + \kappa_{\max} v_T \right]. \quad (\text{B.5})$$

Observe also that the martingale structure of the ε_t 's, Assumption E and (6.1) yield that

$$\begin{aligned} \mathbb{E} \left[\max_{\kappa \in \mathcal{K}_T} \frac{\|\Psi_\kappa \Omega^{-1} \varepsilon\|^2}{T\kappa} \right] &\leq \sum_{\kappa \in \mathcal{K}_T} \frac{1}{T\kappa} \sum_{k=1}^{\kappa} \mathbb{E} \left(\sum_{t=1}^T \frac{\psi_k(X_t) \varepsilon_t}{\sigma^2(X_t)} \right)^2 \leq C \text{Card} \mathcal{K}_T, \\ \mathbb{E} \left[\max_{\kappa \in \mathcal{K}_T} \frac{\|\Psi_\kappa \Omega^{-1} m_T - \mathbb{E} [\Psi_\kappa \Omega^{-1} m_T]\|^2}{T\kappa} \right] &\leq \sum_{\kappa \in \mathcal{K}_T} \frac{1}{T\kappa} \sum_{k=1}^{\kappa} \text{Var} \left(\sum_{t=1}^T \frac{m_T(X_t) \psi_k(X_t)}{\sigma^2(X_t)} \right)^2 \\ &\leq C \text{Card} \mathcal{K}_T \sup_{x \in \Lambda} |m_T(x)|^2. \end{aligned}$$

It follows that

$$\max_{\kappa \in \mathcal{K}_T} \frac{\|\Psi_\kappa \Omega^{-1} \varepsilon\|}{\sqrt{T\kappa}} = O_{\mathbb{P}} \left(\text{Card}^{1/2} \mathcal{K}_T \right), \quad \max_{\kappa \in \mathcal{K}_T} \frac{\|\Psi_\kappa \Omega^{-1} m_T - \mathbb{E} [\Psi_\kappa \Omega^{-1} m_T]\|}{\sqrt{T\kappa}} = O_{\mathbb{P}} \left(\text{Card}^{1/2} \mathcal{K}_T \right) \sup_{x \in \Lambda} |m_T(x)|. \quad (\text{B.6})$$

Note that (A.2) is due to Cauchy-Schwarz inequality and $\left\| \widehat{\Omega}^{-1/2} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1/2} \right\| = 1$. (A.3) follows from (B.3) and (B.6). We now prove (A.4). We have

$$\begin{aligned} \varepsilon' \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon &= \frac{(\varepsilon' \Omega^{-1} \Psi_\kappa + e'_\kappa(\varepsilon)) (\Sigma_\kappa^{-1} + e_\kappa(\Sigma)) (\Psi'_\kappa \Omega^{-1} \varepsilon + e_\kappa(\varepsilon))}{T} \\ &= \frac{\varepsilon' \Omega^{-1} \Psi_\kappa \Sigma_\kappa^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon}{T} + \frac{2\varepsilon' \Omega^{-1} \Psi_\kappa \Sigma_\kappa^{-1} e_\kappa(\varepsilon) + \varepsilon' \Omega^{-1} \Psi_\kappa e_\kappa(\Sigma) \Psi'_\kappa \Omega^{-1} \varepsilon}{T} \\ &\quad + \frac{2\varepsilon' \Omega^{-1} \Psi_\kappa e_\kappa(\Sigma) e_\kappa(\varepsilon) + e'_\kappa(\varepsilon) \Sigma_\kappa^{-1} e_\kappa(\varepsilon) + e'_\kappa(\varepsilon) e_\kappa(\Sigma) e_\kappa(\varepsilon)}{T}. \end{aligned}$$

By (B.3), (B.5), (B.6) and Lemma A.1-(i), Assumption V, $\kappa_{\max} = o(T^{1/3}/\ln^{2/3} T)$ and $\text{Card} \mathcal{K}_T = O(\ln T)$, we have

$$\max_{\kappa \in \mathcal{K}_T} \frac{|\varepsilon' \Omega^{-1} \Psi_\kappa e_\kappa(\Sigma) \Psi'_\kappa \Omega^{-1} \varepsilon|}{T\kappa^{1/2}} \leq \kappa_{\max}^{1/2} \max_{\kappa \in \mathcal{K}_T} \frac{|\varepsilon' \Omega^{-1} \Psi_\kappa e_\kappa(\Sigma) \Psi'_\kappa \Omega^{-1} \varepsilon|}{T\kappa} \leq \kappa_{\max}^{1/2} \max_{\kappa \in \mathcal{K}_T} \frac{\|e_\kappa(\Sigma)\| \|\Psi'_\kappa \Omega^{-1} \varepsilon\|^2}{T\kappa}$$

$$\begin{aligned}
&\leq \kappa_{\max}^{1/2} \max_{\kappa \in \mathcal{K}_T} \frac{\|\Psi'_\kappa \Omega^{-1} \varepsilon\|^2}{T\kappa} \times \max_{\kappa \in \mathcal{K}_T} \|e_\kappa(\Sigma)\| \\
&= O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^3}{T} \right)^{1/2} + \kappa_{\max}^{3/2} v_T \right] \text{Card} \mathcal{K}_T = o_{\mathbb{P}}(1), \\
\max_{\kappa \in \mathcal{K}_T} \frac{|\varepsilon' \Omega^{-1} \Psi_\kappa \Sigma_\kappa^{-1} e_\kappa(\varepsilon)|}{T\kappa^{1/2}} &= O_{\mathbb{P}}(\kappa_{\max}^{1/2}) \max_{\kappa \in \mathcal{K}_T} \frac{\|\varepsilon' \Omega^{-1} \Psi_\kappa\|}{\sqrt{T\kappa}} \times \max_{\kappa \in \mathcal{K}_T} \frac{\|e_\kappa(\varepsilon)\|}{\sqrt{T\kappa}} \\
&= O_{\mathbb{P}}(\kappa_{\max} v_T \text{Card} \mathcal{K}_T),
\end{aligned}$$

the other remainder terms being negligible. This gives (A.3).

We now turn to (A.5). Let $\pi_\kappa(\cdot) = \pi_{\kappa, T}(\cdot)$ be a trigonometric polynomial function of Π_κ with $\sup_{x \in \Lambda} |m_T(x) - \pi_\kappa(x)| \leq 2 \inf_{\pi(\cdot) \in \Pi_\kappa} \sup_{x \in \Lambda} |m_T(x) - \pi(x)|$. Since $\widehat{\Omega}^{-1/2} \pi_{\kappa_{\min}}$ is a linear combination of the columns of $\widehat{\Omega}^{-1/2} \Psi_\kappa$ for all $\kappa \geq \kappa_{\min}$, it follows that $\pi'_{\kappa_{\min}} \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon = \pi'_{\kappa_{\min}} \widehat{\Omega}^{-1} \varepsilon$. This gives

$$m'_T \widehat{\Omega}^{-1} \Psi_\kappa \left(\Psi'_\kappa \widehat{\Omega}^{-1} \Psi_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon = \pi'_{\kappa_{\min}} \widehat{\Omega}^{-1} \varepsilon + (m_T - \pi_{\kappa_{\min}})' \widehat{\Omega}^{-1} \Psi_\kappa \left(T \widehat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon \quad (\text{B.7})$$

with

$$\begin{aligned}
(m_T - \pi_{\kappa_{\min}})' \widehat{\Omega}^{-1} \Psi_\kappa \left(T \widehat{\Sigma}_\kappa \right)^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon &= \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_\kappa}{T} \right] \Sigma_\kappa^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon \\
&+ \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa_{\min}}}{T} \right] \left(\widehat{\Sigma}_\kappa^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} - \Sigma_\kappa^{-1} \Psi'_\kappa \Omega^{-1} \right) \varepsilon \\
&+ \left(\frac{(m_T - \pi_{\kappa_{\min}})' \widehat{\Omega}^{-1} \Psi_\kappa}{T} - \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_\kappa}{T} \right] \right) \widehat{\Sigma}_\kappa^{-1} \Psi'_\kappa \widehat{\Omega}^{-1} \varepsilon.
\end{aligned} \quad (\text{B.8})$$

Consider first the leading term $\pi'_{\kappa_{\min}} \widehat{\Omega}^{-1} \varepsilon$ of (B.7). Because $\sup_{\kappa_{\min}} \sup_{x \in \Lambda} |\pi_{\kappa_{\min}}(x)| < \infty$ and taking $\psi_1(\cdot) = 1$ gives, in (B.2),

$$\left| \pi'_{\kappa_{\min}} \left(\widehat{\Omega}^{-1} - \Omega^{-1} \right) \varepsilon \right| = O_{\mathbb{P}} \left(T^{1/2} v_T \right) \sup_{x \in \Lambda} |\pi_{\kappa_{\min}}(x)| = O_{\mathbb{P}} \left(T^{1/2} v_T \right) \left(\sup_{x \in \Lambda} |m_T(x)| + \sup_{x \in \Lambda} |m_T(x) - \pi_{\kappa_{\min}}(x)| \right).$$

The definition of $\pi_{\kappa_{\min}}(\cdot)$ yields, under Assumption E,

$$\mathbb{E} \left(\pi'_{\kappa_{\min}} \Omega^{-1} \varepsilon \right)^2 = \mathbb{E} \left(\pi'_{\kappa_{\min}} \Omega^{-1} \pi_{\kappa_{\min}} \right) = T \mathbb{E} \left(\frac{\pi_{\kappa_{\min}}(X_t)}{\sigma(X_t)} \right)^2 \leq T \left(\mathbb{E}^{1/2} \left(\frac{\pi_{\kappa_{\min}}(X_t)}{\sigma(X_t)} \right)^2 + \sup_{x \in \Lambda} |m_T(x) - \pi_{\kappa_{\min}}(x)| \right)^2.$$

This gives, for the leading term of (B.7),

$$\max_{\kappa \in \mathcal{K}_T} \left| \frac{\pi'_{\kappa_{\min}} \widehat{\Omega}^{-1} \varepsilon}{\kappa^{1/2}} \right| = \frac{T^{1/2}}{\kappa_{\min}^{1/2}} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{m_T(X_t)}{\sigma(X_t)} \right)^2 + (1 + v_T) \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| + v_T \sup_{x \in \Lambda} |m_T(x)| \right].$$

For the first item of (B.8), note that Assumption E gives that $\text{Var}(\Psi'_\kappa \Omega^{-1} \varepsilon) = T \Sigma_\kappa = \mathbb{E} [\Psi'_\kappa \Omega^{-1} \Psi_\kappa]$, see (6.2). Since orthogonal projection decreases the mean squared norm, this gives, for the first term in (B.8),

$$\begin{aligned}
&\mathbb{E} \left(\mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_\kappa}{T} \right] \Sigma_\kappa^{-1} \Psi'_\kappa \Omega^{-1} \varepsilon \right)^2 \\
&= \frac{1}{T} \mathbb{E} \left[(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_\kappa \right] \left[\mathbb{E} (\Psi'_\kappa \Omega^{-1} \Psi_\kappa) \right]^{-1} \mathbb{E} [\Psi'_\kappa \Omega^{-1} (m_T - \pi_{\kappa_{\min}})] \\
&\leq \frac{1}{T} \mathbb{E} \left[(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} (m_T - \pi_{\kappa_{\min}}) \right] = \mathbb{E} \left(\frac{m_T(X_t) - \pi_{\kappa_{\min}}(X_t)}{\sigma(X_t)} \right)^2 \leq C \sup_{x \in \Lambda} |m_T(x) - \pi_{\kappa_{\min}}(x)|,
\end{aligned}$$

so that

$$\max_{\kappa \in \mathcal{K}_T} \left| \frac{1}{T\kappa^{1/2}} \mathbb{E} \left[(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa} \Sigma_{\kappa}^{-1} \Psi_{\kappa}' \Omega^{-1} \varepsilon \right] \right| = \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| O_{\mathbb{P}} \left(\frac{\text{Card}^{1/2} \mathcal{K}_T}{\kappa_{\min}^{1/2}} \right).$$

For the second term in (B.8), observe that,

$$\begin{aligned} & \left\| \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa}}{T^{1/2}} \right]' \right\|^2 \leq C \left\| \Sigma_{\kappa}^{-1/2} \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa}}{T^{1/2}} \right]' \right\|^2 \\ &= \frac{C}{T} \mathbb{E} \left[(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa} \left[\mathbb{E} (\Psi_{\kappa}' \Omega^{-1} \Psi_{\kappa}) \right]^{-1} \mathbb{E} [\Psi_{\kappa}' \Omega^{-1} (m_T - \pi_{\kappa_{\min}})] \right] \leq C \mathbb{E} \left(\frac{m_T(X_t) - \pi_{\kappa_{\min}}(X_t)}{\sigma(X_t)} \right)^2 \\ &\leq C \sup_{x \in \Lambda} |m_T(x) - \pi_{\kappa_{\min}}(x)|, \text{ and then} \\ & \left| \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa_{\min}}}{T} \right] \left(\widehat{\Sigma}_{\kappa}^{-1} \Psi_{\kappa}' \widehat{\Omega}^{-1} - \Sigma_{\kappa}^{-1} \Psi_{\kappa}' \Omega^{-1} \right) \varepsilon \right| \\ &\leq \frac{C \sup_{x \in \Lambda} |m_T(x) - \pi_{\kappa_{\min}}(x)|}{T^{1/2}} \left[\left\| \widehat{\Sigma}_{\kappa}^{-1} - \Sigma_{\kappa}^{-1} \right\| \left\| \Psi_{\kappa}' \Omega^{-1} \varepsilon \right\| + \left\| \widehat{\Sigma}_{\kappa}^{-1} \right\| \left\| e_{\kappa}(\varepsilon) \right\| \right]. \end{aligned}$$

Therefore Lemma A.1, (B.3) and (B.6) yield

$$\begin{aligned} & \max_{\kappa \in \mathcal{K}_T} \frac{1}{\kappa^{1/2}} \left| \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa_{\min}}}{T} \right] \left(\widehat{\Sigma}_{\kappa}^{-1} \Psi_{\kappa}' \widehat{\Omega}^{-1} - \Sigma_{\kappa}^{-1} \Psi_{\kappa}' \Omega^{-1} \right) \varepsilon \right| \\ &= \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| O_{\mathbb{P}} \left[\left(\frac{\kappa_{\max}^2}{T} \right)^{1/2} + \kappa_{\max} v_T \right] \text{Card}^{1/2} \mathcal{K}_T. \end{aligned}$$

For the last item of (B.8), (B.3), (B.4) and (B.6), Lemma A.1, give that

$$\begin{aligned} & \max_{\kappa \in \mathcal{K}_T} \frac{1}{\sqrt{\kappa}} \left| \left(\frac{(m_T - \pi_{\kappa_{\min}})' \widehat{\Omega}^{-1} \Psi_{\kappa}}{T} - \mathbb{E} \left[\frac{(m_T - \pi_{\kappa_{\min}})' \Omega^{-1} \Psi_{\kappa}}{T} \right] \right) \widehat{\Sigma}_{\kappa}^{-1} \Psi_{\kappa}' \widehat{\Omega}^{-1} \varepsilon \right| \\ &= \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| \kappa_{\max}^{1/2} O_{\mathbb{P}} \left(v_T \text{Card}^{1/2} \mathcal{K}_T \right) \times \sqrt{T \kappa_{\max}} O_{\mathbb{P}} \left[(1 + v_T) \text{Card}^{1/2} \mathcal{K}_T \right] \\ &= \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| T^{1/2} O_{\mathbb{P}} \left(\kappa_{\max}^{1/2} v_T \text{Card} \mathcal{K}_T \right). \end{aligned}$$

Substituting in (B.8) and (B.7) yields

$$\begin{aligned} & \max_{\kappa \in \mathcal{K}_T} \frac{\left| m_T' \widehat{\Omega}^{-1} \Psi_{\kappa} \left(\Psi_{\kappa}' \widehat{\Omega}^{-1} \Psi_{\kappa} \right)^{-1} \Psi_{\kappa}' \widehat{\Omega}^{-1} \varepsilon \right|}{\kappa^{1/2}} \\ &= \frac{T^{1/2}}{\kappa_{\min}^{1/2}} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{m_T(X_t)}{\sigma(X_t)} \right)^2 + (1 + v_T) \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| + v_T \sup_{x \in \Lambda} |m_T(x)| \right] \\ &+ \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| O_{\mathbb{P}} \left[\left(\kappa_{\min}^{-1/2} + \left(\frac{\kappa_{\max}^2}{T} \right)^{1/2} + \kappa_{\max} v_T \right) \text{Card}^{1/2} \mathcal{K}_T + T^{1/2} \kappa_{\max}^{1/2} v_T \text{Card} \mathcal{K}_T \right] \\ &= \frac{T^{1/2}}{\kappa_{\min}^{1/2}} O_{\mathbb{P}} \left[\mathbb{E}^{1/2} \left(\frac{m_T(X_t)}{\sigma(X_t)} \right)^2 + \inf_{\pi(\cdot) \in \Pi_{\kappa_{\min}}} \sup_{x \in \Lambda} |m_T(x) - \pi(x)| + v_T \sup_{x \in \Lambda} |m_T(x)| \right]. \square \end{aligned}$$

B.3 Proof of Lemma A.3

Abbreviate $V_{\kappa}^{-1/2} \Phi_{\kappa}'(X_t) \varepsilon_t$ into η_t . Consider a sequence $\{\tilde{\eta}_t\}_{t \in \mathbb{N}}$ of iid $N(0, \text{Id}_{\kappa})$ variables independent of $\{\varepsilon_t\}_{t \in \mathbb{N}}$ and $\{X_t\}_{t \in \mathbb{N}}$, where Id_{κ} is the identity matrix of dimension $\kappa \times \kappa$. Let $\mathcal{I}(\cdot)$ be a three time differentiable real function.

Define $\tilde{S}_{t+1}^T = \sum_{i=t+1}^T \tilde{\eta}_i$, $\tilde{Q}_T = (T^{-1} \|\tilde{S}_1^T\|^2 - \sigma^2 \kappa) / \sqrt{2\kappa}$. The proof of Lemma A.3 is divided in three steps. The main step aims to establish that for $C(\mathcal{I}) = \max(1, \sup_{z \in \mathbb{R}} |\mathcal{I}'(z)|, \sup_{z \in \mathbb{R}} |\mathcal{I}''(z)|)$ and some $C > 0$ independent of κ and T ,

$$\left| \mathbb{E} [\mathcal{I}(Q_T)] - \mathbb{E} [\mathcal{I}(\tilde{Q}_T)] \right| \leq C \cdot C(\mathcal{I}) \cdot \varphi_\infty^6 \kappa^2 T^{-\frac{3}{2} - \frac{1+a}{5+3a}}. \quad (\text{B.9})$$

Step 1. Proof of (B.9). We build on arguments used in the proof of the Lindeberg Central Limit Theorem as given in Billingsley (1968, Theorem 7.2), see Horowitz and Spokoiny (2001, Lemma 10) for a similar approach in the context of adaptive testing. It consists into successive changes of the η_t into their Gaussian counterparts $\tilde{\eta}_t$, as seen from (B.10) below. However, a important difference is due to the use of nonparametric series methods and dependence. Define

$$S_{tT}(\eta) = S_{t-1} + \eta + \tilde{S}_{t+1}^T, \quad Q_{tT}(\eta) = \frac{T^{-1} S'_{tT}(\eta) S_{tT}(\eta) - \kappa}{\sqrt{2\kappa}}, \quad \mathcal{J}_{tT}(\eta) = \mathcal{I}(Q_{tT}(\eta)) \quad \text{for } \eta \in \mathbb{R}^\kappa.$$

This gives

$$\begin{aligned} \left| \mathbb{E} [\mathcal{I}(Q_T)] - \mathbb{E} [\mathcal{I}(\tilde{Q}_T)] \right| &= \left| \mathbb{E} [\mathcal{J}_{TT}(\eta_T)] - \mathbb{E} [\mathcal{J}_{1T}(\tilde{\eta}_1)] \right| = \left| \sum_{t=1}^T (\mathbb{E} [\mathcal{J}_{tT}(\eta_t)] - \mathbb{E} [\mathcal{J}_{tT}(\tilde{\eta}_t)]) \right| \\ &\leq \sum_{t=1}^T |\mathbb{E} [\mathcal{J}_{tT}(\eta_t)] - \mathbb{E} [\mathcal{J}_{tT}(\tilde{\eta}_t)]|. \end{aligned} \quad (\text{B.10})$$

Define, for $z \in \mathbb{R}$ and $\eta \in \mathbb{R}^\kappa$, $\mathcal{J}_{tT}(z; \eta) = \mathcal{J}_{tT}(z\eta)$. A third-order Taylor expansion of $\mathcal{J}_{tT}(z; \eta)$ with integral remainder yields

$$\mathcal{J}_{tT}(\eta_t) - \mathcal{J}_{tT}(0) = \frac{d\mathcal{J}_{tT}(0; \eta_t)}{dz} + \frac{1}{2} \frac{d^2 \mathcal{J}_{tT}(0; \eta_t)}{d^2 z} + \int_0^1 \frac{(1-z)^2}{2} \frac{d^3 \mathcal{J}_{tT}(z; \eta_t)}{d^3 z} dz,$$

with

$$\begin{cases} \frac{d\mathcal{J}_{tT}(0; \eta_t)}{dz} = \frac{2}{T\sqrt{2\kappa}} \eta'_t S_{tT}(0) \mathcal{I}'(Q_{tT}(0)), \\ \frac{d^2 \mathcal{J}_{tT}(0; \eta_t)}{d^2 z} = \frac{2}{T\sqrt{2\kappa}} \|\eta_t\|^2 \mathcal{I}''(Q_{tT}(0)) + \frac{4}{T^2 \kappa} (\eta'_t S_{tT}(0))^2 \mathcal{I}''(Q_{tT}(0)), \\ \frac{d^3 \mathcal{J}_{tT}(z; \eta_t)}{d^3 z} = \frac{10}{T^2 \kappa} \|\eta_t\|^2 \eta'_t S_{tT}(z\eta_t) \mathcal{I}'''(Q_{tT}(z\eta_t)) + \frac{8}{T^3 \kappa^{3/2}} (\eta'_t S_{tT}(z\eta_t))^3 \mathcal{I}'''(Q_{tT}(z\eta_t)). \end{cases} \quad (\text{B.11})$$

Let $\tilde{\mathcal{F}}_t$ be the sigma-field generated by $\dots, \eta_{t-2}, \eta_{t-1}, \tilde{\eta}_{t+1}, \tilde{\eta}_{t+2}, \dots$ and note that $S_{tT}(0)$ and $Q_{tT}(0)$ are $\tilde{\mathcal{F}}_t$ -measurable. Since η_t and $\tilde{\eta}_t$ are centered given $\tilde{\mathcal{F}}_t$, we have

$$\mathbb{E} \left[\frac{d\mathcal{J}_{tT}(0; \eta_t)}{dz} - \frac{d\mathcal{J}_{tT}(0; \tilde{\eta}_t)}{dz} \right] = \mathbb{E} \left[\frac{2}{T\sqrt{2\kappa}} S'_{tT}(0) \mathcal{I}'(Q_{tT}(0)) \mathbb{E} [(\eta_t - \tilde{\eta}_t) | \tilde{\mathcal{F}}_t] \right] = 0,$$

and substituting the Taylor expansion in (B.10) yields

$$\left| \mathbb{E} [\mathcal{I}(Q_T)] - \mathbb{E} [\mathcal{I}(\tilde{Q}_T)] \right| \leq \frac{1}{2} \sum_{t=1}^T \left| \mathbb{E} \left[\frac{d^2 \mathcal{J}_{tT}(0; \eta_t)}{d^2 z} - \frac{d^2 \mathcal{J}_{tT}(0; \tilde{\eta}_t)}{d^2 z} \right] \right| \quad (\text{B.12})$$

$$+ \frac{1}{2} \sum_{t=1}^T \int_0^1 (1-z)^2 \left[\left| \mathbb{E} \frac{d^3 \mathcal{J}_{tT}(z; \eta_t)}{d^3 z} \right| + \left| \mathbb{E} \frac{d^3 \mathcal{J}_{tT}(z; \tilde{\eta}_t)}{d^3 z} \right| \right] dz, \quad (\text{B.13})$$

and we now bound each of these two sums.

We begin by establishing a preliminary inequality. Let n_1 and n_2 be two positive real numbers with $2 \leq n_1 + n_2 \geq 8$. Then for any t, t' and $z \in [0, 1]$,

$$\max(\mathbb{E}\|S_{tT}(z\eta_t)\|^{n_1}\|\eta_{t'}\|^{n_2}, \mathbb{E}\|S_{tT}(z\tilde{\eta}_t)\|^{n_1}\|\tilde{\eta}_{t'}\|^{n_2}) \leq C\varphi_\infty^{n_1+n_2}\kappa^{\frac{n_1+n_2}{2}}T^{\frac{n_1}{2}}. \quad (\text{B.14})$$

We give a proof for $\mathbb{E}\|S_{tT}(z\eta_t)\|^{n_1}\|\eta_{t'}\|^{n_2}$, the other bound being similarly established. The Hölder inequality implies that

$$\begin{aligned} \mathbb{E}\|S_{tT}(z\eta_t)\|^{n_1}\|\eta_{t'}\|^{n_2} &\leq \mathbb{E}^{\frac{n_1}{n_1+n_2}}\|S_{tT}(z\eta_t)\|^{n_1+n_2}\mathbb{E}^{\frac{n_2}{n_1+n_2}}\|\eta_{t'}\|^{n_1+n_2} \\ &\leq \mathbb{E}^{\frac{n_1}{n_1+n_2}}\left(\|S_t + z\eta_t\| + \|\tilde{S}_{t+1}^T\|\right)^{n_1+n_2} \times \left(\|V_\kappa^{-1/2}\|^{n_2}\mathbb{E}^{\frac{n_2}{n_1+n_2}}\left[\sum_{k=1}^\kappa\varphi_k^2(X_{t'})\varepsilon_{t'}^2\right]^{\frac{n_1+n_2}{2}}\right) \\ &\leq 2^{n_1+n_2-1}\left(\mathbb{E}^{\frac{n_1}{n_1+n_2}}\|S_t + z\eta_t\|^{n_1+n_2} + \mathbb{E}^{\frac{n_1}{n_1+n_2}}\|\tilde{S}_{t+1}^T\|^{n_1+n_2}\right)\left(\|V_\kappa^{-1/2}\|^{n_2}\varphi_\infty^{n_2}\kappa^{n_2/2}\mathbb{E}^{\frac{n_2}{n_1+n_2}}|\varepsilon_{t'}|^{n_1+n_2}\right). \end{aligned}$$

Because $\tilde{\eta}_t$ is a $N(0, \sigma^2\text{Id}_\kappa)$, it is easily seen that $\mathbb{E}^{\frac{n_1}{n_1+n_2}}\|\tilde{S}_{t+1}^T\|^{n_1+n_2} \leq C\sigma^{n_1+n_2}(\kappa T)^{n_1/2}$ and we now bound $\mathbb{E}^{\frac{n_1}{n_1+n_2}}\|S_t + z\eta_t\|^{n_1+n_2}$. We have, by convexity, the Burkholder inequality (see Chow and Teicher, 1988, p. 396, noticing that $\sum_{i=1}^{t-1}\varphi_k(X_i)\varepsilon_i + z\varphi_k(X_t)\varepsilon_t$ is a sum of difference of martingale), and the Minkowski inequality.

$$\begin{aligned} \mathbb{E}\|S_t + z\eta_t\|^{n_1+n_2} &\leq \|V_\kappa^{-1/2}\|^{n_1+n_2}\kappa^{\frac{n_1+n_2}{2}}\mathbb{E}\left[\frac{1}{\kappa}\sum_{k=1}^\kappa\left(\sum_{i=1}^{t-1}\varphi_k(X_i)\varepsilon_i + z\varphi_k(X_t)\varepsilon_t\right)^2\right]^{\frac{n_1+n_2}{2}} \\ &\leq \|V_\kappa^{-1/2}\|^{n_1+n_2}\kappa^{\frac{n_1+n_2}{2}-1}\sum_{k=1}^\kappa\left[\mathbb{E}^{\frac{1}{n_1+n_2}}\left|\sum_{i=1}^{t-1}\varphi_k(X_i)\varepsilon_i + z\varphi_k(X_t)\varepsilon_t\right|^{n_1+n_2}\right]^{n_1+n_2} \\ &\leq \|V_\kappa^{-1/2}\|^{n_1+n_2}\kappa^{\frac{n_1+n_2}{2}-1}C\sum_{k=1}^\kappa\left[\mathbb{E}^{\frac{2}{n_1+n_2}}\left(\sum_{i=1}^{t-1}\varphi_k^2(X_i)\varepsilon_i^2 + z^2\varphi_k^2(X_t)^2\varepsilon_t^2\right)^{\frac{n_1+n_2}{2}}\right]^{\frac{n_1+n_2}{2}} \\ &\leq C\|V_\kappa^{-1/2}\|^{n_1+n_2}\kappa^{\frac{n_1+n_2}{2}-1}\sum_{k=1}^\kappa\left[\sum_{i=1}^t\mathbb{E}^{\frac{2}{n_1+n_2}}|\varphi_k^2(X_i)\varepsilon_i^2|^{\frac{n_1+n_2}{2}}\right]^{\frac{n_1+n_2}{2}} \\ &\leq C\|V_\kappa^{-1/2}\|^{n_1+n_2}(\kappa T)^{\frac{n_1+n_2}{2}}\varphi_\infty^{n_1+n_2}\mathbb{E}|\varepsilon_t|^{n_1+n_2}. \end{aligned}$$

This gives $\mathbb{E}^{\frac{n_1}{n_1+n_2}}\|S_t + z\eta_t\|^{n_1+n_2} \leq C(\kappa T)^{n_1/2}\varphi_\infty^{n_1}$ and then (B.14).

We now return to (B.13). The expression (B.11) of the third derivative of $\mathcal{J}_{tT}(z; \eta_t)$ and (B.14) yield

$$\begin{aligned} &\sum_{t=1}^T\int_0^1(1-z)^2\left[\left|\mathbb{E}\frac{d^3\mathcal{J}_{tT}(z; \eta_t)}{d^3z}\right| + \left|\mathbb{E}\frac{d^3\mathcal{J}_{tT}(z; \tilde{\eta}_t)}{d^3z}\right|\right]dz \\ &\leq C(\mathcal{I})\sum_{t=1}^T\int_0^1(1-z)^2\left\{\frac{10}{T^2\kappa}\sum_{t=1}^T\mathbb{E}\left[\|\eta_t\|^3\|S_{tT}(z\eta_t)\| + \|\tilde{\eta}_t\|^3\|S_{tT}(z\tilde{\eta}_t)\|\right] \right. \\ &\quad \left. + \frac{8}{T^3\kappa^{3/2}}\mathbb{E}\left[\|\eta_t\|^3\|S_{tT}(z\eta_t)\|^3 + \|\tilde{\eta}_t\|^3\|S_{tT}(z\tilde{\eta}_t)\|^3\right]\right\}dz \\ &\leq C\varphi_\infty^6.C(\mathcal{I})\left(\kappa.T^{-1/2} + \kappa^{3/2}.T^{-1/2}\right) \leq C\varphi_\infty^6.C(\mathcal{I})\left(\frac{\kappa^3}{T}\right)^{1/2}. \quad (\text{B.15}) \end{aligned}$$

To study (B.12), let $\bar{\Phi}_\kappa(X_t) = V_\kappa^{-1/2} \Phi'_\kappa(X_t) = [\bar{\varphi}_1(X_t), \dots, \bar{\varphi}_\kappa(X_t)]'$, $S_{tT} = S_{tT}(0) = [S_{1tT}, \dots, S_{\kappa tT}]'$, $Q_{tT} = Q_{tT}(0)$. The definitions of η_t , $\tilde{\eta}_t$ and $\tilde{\mathcal{F}}_t$ shows that $\mathbb{E}[\eta_{kt}\eta_{\ell t} - \tilde{\eta}_{kt}\tilde{\eta}_{\ell t} | \tilde{\mathcal{F}}_t] = \bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t) - \mathbb{I}(\kappa = \ell) = \bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t) - \mathbb{E}[\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t)]$. Therefore since Q_{tT} and ${}_{tT}$ are $\tilde{\mathcal{F}}_t$ measurable, conditioning with respect to $\tilde{\mathcal{F}}_t$ yields, using the expression of the second order derivative of $\mathcal{J}_{tT}(0; \eta_t)$ given in (B.11)

$$\begin{aligned} \mathbb{E} \left[\frac{d^2 \mathcal{J}_{tT}(0; \eta_t)}{d^2 z} - \frac{d^2 \mathcal{J}_{tT}(0; \tilde{\eta}_t)}{d^2 z} \right] &= \frac{2}{T\sqrt{2\kappa}} \sum_{k=1}^{\kappa} \mathbb{E} \left[(\bar{\varphi}_k^2(X_t) - \mathbb{E}\bar{\varphi}_k^2(X_t)) \mathcal{I}'(Q_{tT}) \right] \\ &+ \frac{4}{T^2\kappa} \sum_{1 \leq k, \ell \leq \kappa} \mathbb{E} \left[(\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t) - \mathbb{E}[\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t)]) S_{ktT} S_{\ell tT} \mathcal{I}''(Q_{tT}) \right] \\ &= \frac{2}{T\sqrt{2\kappa}} \sum_{k=1}^{\kappa} \text{Cov}(\bar{\varphi}_k^2(X_t), \mathcal{I}'(Q_{tT})) \\ &+ \frac{4}{T^2\kappa} \sum_{1 \leq k, \ell \leq \kappa} \text{Cov}(\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t), S_{ktT} S_{\ell tT} \mathcal{I}''(Q_{tT})) . \end{aligned}$$

Let n be an integer and define

$$\check{S}_{tT} = S_{t-n-1} + \tilde{S}_{t+1}^T, \check{Q}_{tT} = \frac{T^{-1} \|\check{S}_{tT}\|^2 - \sigma^2 \kappa}{\sqrt{2\kappa}} = Q_{tT} - \frac{2\check{S}_{tT}'(S_t - S_{t-n-1}) + \|S_t - S_{t-n-1}\|^2}{T\sqrt{2\kappa}} .$$

The variable \check{Q}_{tT} and \check{S}_{tT} depend upon $\tilde{\eta}_{t+1}, \dots, \tilde{\eta}_T$ and $\eta_1, \dots, \eta_{t-n-1}$, which are $n+1$ time periods far from the $\bar{\varphi}_k^2(X_t)$'s. Since $\sup_{x \in \mathbb{R}^d} |\bar{\varphi}_k(x)| \leq \sup_{x \in \mathbb{R}^d} \|\bar{\Phi}_\kappa(x)\| \leq \|V_\kappa^{-1/2}\| \varphi_\infty \sqrt{\kappa}$, the Wolkonski-Rozanov inequality yields

$$\left| \sum_{k=1}^{\kappa} \text{Cov}(\bar{\varphi}_k^2(X_t), \mathcal{I}'(\check{Q}_{tT})) \right| \leq 4C(\mathcal{I}) \|V_\kappa^{-1/2}\|^2 \varphi_\infty^2 \kappa \alpha(n), \quad (\text{B.16})$$

$$\begin{aligned} \left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov}[\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t), \check{S}_{ktT} \check{S}_{\ell tT} \mathcal{I}''(\check{Q}_{tT})] \right| &\leq 8C(\mathcal{I}) \|V_\kappa^{-1/2}\|^2 \varphi_\infty^2 \kappa \alpha^{3/4}(n) \sum_{1 \leq k, \ell \leq \kappa} \mathbb{E}^{\frac{1}{4}} |\check{S}_{ktT} \check{S}_{\ell tT}|^4 \\ &\leq 8C(\mathcal{I}) \|V_\kappa^{-1/2}\|^2 \varphi_\infty^2 \kappa \alpha^{3/4}(n) \sum_{1 \leq k, \ell \leq \kappa} \mathbb{E}^{\frac{1}{8}} |\check{S}_{ktT}|^8 \mathbb{E}^{\frac{1}{8}} |\check{S}_{\ell tT}|^8 \\ &\leq C.C(\mathcal{I}) \|V_\kappa^{-1/2}\|^4 \varphi_\infty^4 \kappa^3 T \alpha^{3/4}(n), \quad (\text{B.17}) \end{aligned}$$

by first integrating out with respect to the $\tilde{\eta}_{t+1}, \dots, \tilde{\eta}_T$, which are independent from the η_t 's, and using (B.14). Note that $\mathbb{E}\bar{\varphi}_k(X_t)^4 \leq \|V_\kappa^{-1/2}\|^2 \varphi_\infty^2 \kappa \mathbb{E}\bar{\varphi}_k(X_t)^2 = \|V_\kappa^{-1/2}\|^2 \varphi_\infty^2 \kappa$ and $\text{Var}^{1/2}(\bar{\varphi}_k^2(X_t)\bar{\varphi}_\ell^2(X_t)) \leq (\mathbb{E}\bar{\varphi}_k^4(X_t)\mathbb{E}\bar{\varphi}_\ell^4(X_t))^{1/4} \leq \|V_\kappa^{-1/2}\| \varphi_\infty \sqrt{\kappa}$. This together with the definition of \check{Q}_{tT} and (B.14) gives

$$\begin{aligned} &\left| \sum_{k=1}^{\kappa} \text{Cov}[\bar{\varphi}_k^2(X_t), \mathcal{I}'(Q_{tT}) - \mathcal{I}'(\check{Q}_{tT})] \right| \\ &\leq \frac{C(\mathcal{I})}{T\sqrt{2\kappa}} \sum_{k=1}^{\kappa} \mathbb{E} \left[|\bar{\varphi}_k^2(X_t) - 1| \left(2\|\check{S}_{tT}\| \|S_{t-1} - S_{t-n-1}\| + \|S_{t-1} - S_{t-n-1}\|^2 \right) \right] \\ &\leq \frac{C(\mathcal{I})}{T\sqrt{2\kappa}} \sum_{k=1}^{\kappa} \mathbb{E}^{1/2} |\bar{\varphi}_k^2(X_t) - 1|^2 \left(\mathbb{E}^{1/4} \|\check{S}_{tT}\|^4 \times \mathbb{E}^{1/4} \|S_{t-1} - S_{t-n-1}\|^4 + \mathbb{E}^{1/2} \|S_{t-1} - S_{t-n-1}\|^4 \right) \\ &\leq C \frac{C(\mathcal{I})}{T\sqrt{2\kappa}} \left(\kappa \times \varphi_\infty \kappa^{1/2} \times \left(\varphi_\infty \sqrt{\kappa T} \varphi_\infty \sqrt{\kappa n} + \varphi_\infty^2 \kappa n \right) \right) = C.C(\mathcal{I}) \cdot \varphi_\infty^3 \kappa^2 \frac{\sqrt{Tn} + n}{T}, \\ &\left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov}[\bar{\varphi}_k(X_t)\bar{\varphi}_\ell(X_t), S_{\ell tT} S_{ktT} (\mathcal{I}''(Q_{tT}) - \mathcal{I}''(\check{Q}_{tT}))] \right| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{C(\mathcal{I})}{T\sqrt{2\kappa}} \left(\mathbb{E}^{1/8} \|\check{S}_{tT}\|^8 \mathbb{E}^{1/8} \|S_{t-1} - S_{t-n-1}\|^8 + \mathbb{E}^{1/4} \|S_{t-1} - S_{t-n-1}\|^8 \right) \\
&\quad \times \sum_{1 \leq k, \ell \leq \kappa} \text{Var}^{1/2} (\bar{\varphi}_k^2(X_t) \bar{\varphi}_\ell^2(X_t)) \mathbb{E}^{1/8} S_{\ell tT}^8 \mathbb{E}^{1/8} S_{ktT}^8 \\
&\leq C \frac{C(\mathcal{I})}{T\sqrt{\kappa}} \kappa \varphi_\infty^2 (\sqrt{Tn} + n) \times \kappa^2 \varphi_\infty \sqrt{\kappa} \varphi_\infty^2 T = C.C(\mathcal{I}).\varphi_\infty^5 \kappa^3 (\sqrt{Tn} + n), \\
&\left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), (S_{\ell tT} - \check{S}_{\ell tT}) (S_{ktT} - \check{S}_{ktT}) \mathcal{I}''(\check{Q}_{tT}) \right] \right| \\
&\leq C(\mathcal{I}) \sum_{1 \leq k, \ell \leq \kappa} \text{Var}^{1/2} (\bar{\varphi}_k^2(X_t) \bar{\varphi}_\ell^2(X_t)) \mathbb{E}^{1/4} (S_{\ell tT} - \check{S}_{\ell tT})^4 \mathbb{E}^{1/4} (S_{ktT} - \check{S}_{ktT})^4 \\
&\leq C.C(\mathcal{I}).\varphi_\infty^3 \kappa^{5/2} n, \\
&\left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), \check{S}_{ktT} (S_{\ell tT} - \check{S}_{\ell tT}) \mathcal{I}''(\check{Q}_{tT}) \right] \right| \\
&\leq C(\mathcal{I}) \sum_{1 \leq k, \ell \leq \kappa} \text{Var}^{1/2} (\bar{\varphi}_k^2(X_t) \bar{\varphi}_\ell^2(X_t)) \mathbb{E}^{1/4} \check{S}_{ktT}^4 \mathbb{E}^{1/4} (S_{\ell tT} - \check{S}_{\ell tT})^4 \\
&\leq C.C(\mathcal{I}).\varphi_\infty^3 \kappa^{5/2} \sqrt{nT}.
\end{aligned}$$

Therefore, (B.16), (B.17) and these inequalities give

$$\begin{aligned}
&\left| \sum_{k=1}^{\kappa} \text{Cov} (\bar{\varphi}_k^2(X_t), \mathcal{I}'(Q_{tT})) \right| \\
&\leq C.C(\mathcal{I}).\varphi_\infty^3 \kappa \left(\alpha_X(n) + \kappa \frac{(\sqrt{Tn} + n)}{T} \right) \\
&\left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} (\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), S_{ktT} S_{\ell tT} \mathcal{I}''(Q_{tT})) \right| \\
&\leq \left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), \check{S}_{ktT} \check{S}_{\ell tT} \mathcal{I}''(\check{Q}_{tT}) \right] \right| \\
&\quad + \left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), S_{ktT} S_{\ell tT} (\mathcal{I}''(Q_{tT}) - \mathcal{I}''(\check{Q}_{tT})) \right] \right| \\
&\quad + 2 \left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), \check{S}_{ktT} (S_{\ell tT} - \check{S}_{\ell tT}) \mathcal{I}''(\check{Q}_{tT}) \right] \right| \\
&\quad + \left| \sum_{1 \leq k, \ell \leq \kappa} \text{Cov} \left[\bar{\varphi}_k(X_t) \bar{\varphi}_\ell(X_t), (S_{\ell tT} - \check{S}_{\ell tT}) (S_{ktT} - \check{S}_{ktT}) \mathcal{I}''(\check{Q}_{tT}) \right] \right| \\
&\leq C.C(\mathcal{I}).\varphi_\infty^5 \kappa^3 \left(T \alpha_X^{3/4}(n) + \sqrt{nT} + n \right).
\end{aligned}$$

Summing over t gives in (B.12)

$$\sum_{t=1}^T \left| \mathbb{E} \left[\frac{d^2 \mathcal{J}_{tT}(0; \eta_t)}{d^2 z} - \frac{d^2 \mathcal{J}_{tT}(0; \tilde{\eta}_t)}{d^2 z} \right] \right|$$

$$\begin{aligned}
&\leq C.C(\mathcal{I}).\varphi_\infty^3\sqrt{\kappa}\left(\alpha_X(n) + \kappa\frac{(\sqrt{Tn} + n)}{T}\right) + C.C(\mathcal{I}).\varphi_\infty^5\frac{\kappa^2}{T}\left(T\alpha_X^{3/4}(n) + \sqrt{nT} + n\right) \\
&\leq C.C(\mathcal{I}).\varphi_\infty^5\left[\sqrt{\kappa}\alpha_X(n) + \kappa^2\alpha_X^{3/4}(n) + (\kappa^{3/2} + \kappa^2)\frac{(\sqrt{Tn} + n)}{T}\right] \\
&\leq C.C(\mathcal{I}).\varphi_\infty^5\kappa^2\left(\alpha_X^{3/4}(n) + \frac{(\sqrt{Tn} + n)}{T}\right) \\
&\leq C.C(\mathcal{I}).\varphi_\infty^5\kappa^2\left(n^{-\frac{3}{4}(1+a)} + \frac{(\sqrt{Tn} + n)}{T}\right), \tag{B.18}
\end{aligned}$$

under Assumption M-i. An optimal choice of the order of n in (B.18) is $T^{2/(5+3a)}$ which gives the upper bound $C.C(\mathcal{I}).\varphi_\infty^5\kappa^2T^{-\frac{3}{2}\frac{1+a}{5+3a}}$. Therefore (B.18) and (B.12), (B.15) and (B.13) yields that (B.9) is proven.

Step 2. Proof of Lemma A.3-i. Choose now a three time continuously differentiable $\mathcal{I}_\epsilon(z)$ with $\mathcal{I}_\epsilon(z) = 0$ if $z \leq -\epsilon$, $\mathcal{I}_\epsilon(z) = 1$ if $z > 0$. This gives, for any $\gamma \in \mathbb{R}$,

$$\mathbb{I}(z \geq \gamma) \leq \mathcal{I}_\epsilon(z - \gamma) \leq \mathbb{I}(z \geq \gamma - \epsilon), \tag{B.19}$$

and then, by (B.9),

$$\begin{aligned}
\mathbb{P}(Q_T \geq \gamma) &\leq \mathbb{E}\mathcal{I}_\epsilon(Q_T - \gamma) \leq \mathbb{E}\mathcal{I}_\epsilon(\tilde{Q}_T - \gamma) + C_\epsilon.\varphi_\infty^6\kappa^2T^{-\frac{3}{2}\frac{1+a}{5+3a}} \\
&\leq \mathbb{P}(\tilde{Q}_T \geq \gamma - \epsilon) + o(1), \\
\mathbb{P}(Q_T \geq \gamma) &\geq \mathbb{E}\mathcal{I}_\epsilon(Q_T - \gamma - \epsilon) \geq \mathbb{E}\mathcal{I}_\epsilon(\tilde{Q}_T - \gamma - \epsilon) + o(1) \\
&\geq \mathbb{P}(\tilde{Q}_T \geq \gamma + \epsilon) + o(1).
\end{aligned} \tag{B.20}$$

Note that \tilde{Q}_T is a $(\chi(\kappa) - \kappa)/\sqrt{2\kappa}$ which has a continuous density and converges in distribution to a standard normal if κ goes to infinity. Therefore taking ϵ small enough gives Lemma A.3-i.

Step 3. Proof of Lemma A.3-ii. The proof is done by bounding $\mathbb{E}\mathcal{I}_\epsilon(\tilde{Q}_T - \gamma)$ in (B.20). Observe that \tilde{Q}_T has the same distribution as

$$\frac{1}{\sqrt{\kappa}} \sum_{k=1}^{\kappa} \frac{\zeta_k^2 - 1}{\sqrt{2}},$$

where the ζ_k 's are iid $N(0, 1)$ random variables. As established in the proof of Theorem 7.2 of Billingsley (1968) and changing the $(\zeta_k^2 - 1)/\sqrt{2}$ into standard $N(0, 1)$ variables, there is a constant C_ϵ with

$$\left| \mathbb{E}\mathcal{I}_\epsilon(\tilde{Q}_T - \gamma) - \mathbb{E}\mathcal{I}_\epsilon(N(0, 1) - \gamma) \right| \leq \frac{C_\epsilon}{\sqrt{\kappa}}.$$

Then (B.19) and (B.20) show $\mathbb{P}(Q_T \geq \gamma) \leq \mathbb{P}(N(0, 1) \geq \gamma - \epsilon) + C_\epsilon \left[\kappa^2 T^{-\frac{3}{2}\frac{1+a}{5+3a}} + \frac{1}{\sqrt{\kappa}} \right]$ Applying the Mill's ratio inequality (see Shorack and Wellner (1986), p.850) to $\mathbb{P}(N(0, 1) \geq \gamma - \epsilon)$ shows that Lemma A.3-ii is proven. \square

Table 1: Size properties (5 %) of our and Hamilton tests (LM)
(200 observations)

		<i>c</i>			LM
distribution	ρ	2	3	5	
normal	0	.057	.048	.047	.047
student	0	.055	.047	.046	.047
exponential	0	.056	.048	.047	.048
normal	.25	.057	.048	.047	.045
student	.25	.059	.052	.051	.051
exponential	.25	.060	.051	.049	.049
normal	.50	.057	.047	.045	.044
student	.50	.056	.050	.050	.051
exponential	.50	.057	.050	.047	.049
normal	.75	.052	.044	.043	.044
student	.75	.059	.051	.050	.050
exponential	.75	.058	.048	.046	.055

Table 2: Power properties (5 %) of our and Hamilton tests (LM): first experiment
(200 observations)

ρ_1	ρ_2	$ \rho_2 - \rho_1 $	c			LM
			2	3	5	
0	.25	.25	.236	.224	.222	.118
	.50	.50	.653	.646	.645	.361
	.75	.75	.849	.846	.846	.682
.25	.5	.25	.237	.227	.226	.128
	.75	.50	.583	.576	.575	.413
	-.50	.75	.947	.945	.944	.666
.50	.25	.25	.261	.247	.246	.123
	0	.50	.725	.715	.713	.360
	-.25	.75	.967	.965	.964	.652
.75	.50	.25	.312	.298	.295	.160
	.25	.50	.797	.785	.781	.411
	0	.75	.979	.976	.975	.673

Table 3: Power properties (5 %) of our and Hamilton tests (LM):second experiment
(200 observations)

		<i>c</i>			LM
τ	ρ	2	3	5	
1	.25	.168	.161	.161	.056
	.50	.426	.421	.420	.072
	.75	.564	.555	.553	.105
.50	.25	.245	.233	.231	.080
	.50	.639	.605	.595	.213
	.75	.758	.716	.699	.477
.25	.25	.301	.263	.254	.278
	.50	.751	.664	.622	.716
	.75	.857	.764	.702	.776

Figure 1

Alternative model ($\rho = .5$)

Dashed line is for $\tau = .25$, thick line for $\tau = .5$ and solid line for $\tau = 1$.

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