ECOM 009 Macroeconomics B

Lecture 1

Giulio Fella

How macroeconomists think

Macroeconomists (and economists in general) think by means of theories or models → Abstract (mathematical) representations of reality.

Method: Scientific and inductive.

1. Set of facts to explain. Accurately **measure** the variables of interest for such facts.
2. Conjecture a **theory** involving the (endogenous) variables whose behaviour has to be explained.

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   - Logical deductions from the assumptions → implications for endogenous variables.
3. Validate or falsify the theory.

How macroeconomists think

Logical consistency of deductions is necessary but not sufficient for a good theory. A good (useful) theory is not rejected by the data. It is better than an alternative one if:

- fits the data better in a statistical sense;
- can explain a larger set of facts.

How macroeconomists think (II)

- A model performs two roles:
  - **Measurement tool**: e.g. How big a fraction of y can be explained by x?
  - **Laboratory for experiments**: e.g. What happens to y if x changes by 1%?

How macroeconomists think (II)

- A model performs two roles:
  - **Measurement tool**: e.g. How big a fraction of y can be explained by x?
  - **Laboratory for experiments**: e.g. What happens to y if x changes by 1%?
- Components of a model.
  - The **model environment**. Exogenous preferences, technology, market structure, property rights.
  - The **model equilibrium** . The *vector* of (values for) endogenous variables which satisfies all the conditions of the model.

Example 1

General equilibrium for a static (just one period) endowment economy. Two agents (A and B) and two goods (1 and 2). Endowments (e_1^A, e_2^A) and (e_1^B, e_2^B).

Planning equilibrium. Vector (list of numbers) of *allocations* (quantities) of goods supplied and demanded that satisfies the technological constraint (i.e. for each good total consumption does not exceed total endowment). In symbols, a vector $\{e_1^A, e_2^A, e_1^B, e_2^B, c_1^A, c_2^A, c_1^B, c_2^B\}$ such that

1. $c_1^A + c_1^B \leq e_1^A + e_1^B$ and $c_2^A + c_2^B \leq e_2^A + e_2^B$.

Example 1 (II)

Market equilibrium. Vector of allocations *and prices* such that consumers maximize their utility and all markets clear (demand equal supply). In symbols, a vector $\{e_1^A, e_2^A, e_1^B, e_2^B, c_1^A, c_2^A, c_1^B, c_2^B, p_1, p_2\}$ such that

1. consumers maximize utility *given prices*;
2. $c_1^A + c_1^B = e_1^A + e_1^B$ and $c_2^A + c_2^B = e_2^A + e_2^B$.

Example 2

General equilibrium for the dynamic (many periods indexed by t) counterpart of the same endowment economy. Endowments $(e_1^A(t), e_2^A(t))$ and $(e_1^B(t), e_2^B(t))$.

The model equilibrium is no longer a *vector* (list of numbers) for the endogenous variables. It is a *vector of functions of time;* i.e. one list of numbers for the endogenous variables for *each time period* (indexed by time).

Example 2 (II)

Planning equilibrium: a vector $\{e_1^A(t), e_2^A(t), e_1^B(t), e_2^B(t), c_1^A(t), c_2^A(t), c_1^B(t), c_2^B(t)\}$ such that

1. $c_1^A(t) + c_1^B(t) \leq e_1^A(t) + e_1^B(t)$ and $c_2^A(t) + c_2^B(t) \leq e_2^A(t) + e_2^B(t)$.

Market equilibrium: a vector $\{e_1^A(t), e_2^A(t), e_1^B(t), e_2^B(t), c_1^A(t), c_2^A(t), c_1^B(t), c_2^B(t), p_1(t), p_2(t)\}$ such that

1. consumers maximize utility given prices and
2. $c_1^A(t) + c_1^B(t) = e_1^A(t) + e_1^B(t)$ and $c_2^A(t) + c_2^B(t) = e_2^A(t) + e_2^B(t)$.

Why time

Modern macroeconomics is inherently dynamics because:

- Present affects the future through accumulation.
- If agents are forward looking, future affects the present through expectations.

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- Present affects the future through accumulation.
- If agents are forward looking, future affects the present through expectations.

→ Nearly all the equilibria we will study in macroeconomics will be *vectors of functions of time.*

Dynamic opt. - finite horizon*

Consider the following, finite horizon, saving problem.

Optimal control or sequence problem (SP)

$$W = \max_{\{c_s, a_{s+1}\}_{s=t}^{T}} \sum_{s=t}^{T} \beta^{s-t} \mathbb{E}_t u(c_s) \tag{1}$$

$$\text{s.t. } a_{s+1} = (1+r)\, a_s + y_s - c_s, \tag{2}$$

$$a_t \text{ given, } a_{T+1} \geq b \tag{3}$$

with u continuous and strictly increasing.

* The following notes have benefitted from Chapter 3 of Per Krusell's very readable lecture notes that you can find here. I have gone slower and easier on the infinite horizon stuff. You may want to take a look at the original notes for a deeper treatement.

Deterministic, finite horizon

Deterministic case: y_s is constant (e.g. $y_s = 0$, for all s).

Solution: a pair of sequences $\{c_s, a_{s+1}\}_{s=t}^{T}$ of real numbers; i.e. two vectors in $\mathbb{R}^{T-t+1}$.

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Existence of a solution: If the choice set is non-empty and compact, a solution exists (Weierstrass theorem). This is the case if b is finite (e.g. solvency $b \geq 0$).

Deterministic, finite horizon (II)

More formally, for given T, b the solution is a pair of sequences $\{c_s(a_t), a_{s+1}(a_t)\}_{s=t}^{T}$ such that the Euler equation

$$u'(c_s) = \beta(1+r)u'(c_{s+1}) \tag{4}$$

holds together with (2) and (3).

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- System of 2 first-order difference equations.
- Need two boundary conditions for a unique solution.
- Otherwise infinite number of solutions to the system of difference equations and no solution (no max) for the optimization problem.

Deterministic, finite horizon (III)

- For given T, b the sequences depend on a_t, t. Hence, the value function W, the maximand evaluated along an optimal path, is also a function $W_t(a_t)$.

Stochastic, finite horizon

Stochastic case: We restrict attention to stochastic process of the form:

$$y_t = g(v_{t-1}, v_{t-2}, ..., v_{t-k}) + u_t$$

with u_t a random innovation.*

- The above restriction implies $\mathbb{E}[y_{t+1}|y_t, v_t, ..., v_{t-k}, v_{t-k-1}, ..., v_0] = E[y_{t+1}|y_t, v_t, ..., v_{t-k+1}]$.
- Therefore the variable $z_t = \{y_t, v_t, ..., v_{t-k+1}\}$ is a conservatively appropriate (i.e correct but possibly redundant) state variable

* The above process encompasses both finite AR and finite MA processes. This can be seen by setting $v_{t-i} = y_{t-i}$ for an AR process and $v_{t-i} = u_{t-i}$ for an MA process.

Stochastic, finite horizon (II)

- The solution is a pair of sequences $\{c_s(a_t, z_t), a_{s+1}(a_t, z_t)\}_{s=t}^{T}$ *for each possible history* $\{z_s\}_{s=t}^{T}$ such that the Euler equation

$$u'(c_s) = \beta(1+r)\mathbb{E}_s u'(c_{s+1}) \tag{5}$$

  holds together with (2) and (3).
- The possible future histories depend on current z_t and therefore so do the optimal sequences and the value function $W_t(a_t, z_t)$.

Stochastic, finite horizon (III)

- As there is more than one possible history (compared to the deterministic case), the solution has now a larger dimension.
- Two issues:
  1. does a solution exist;
  2. is the SP the easiest method to find it.

1. Does a solution exist?

Two possible cases:

a) v_s, hence y_s, can take only a finite number of values. We are still dealing with vectors in $\mathbb{R}^h$ for some h and Weierstrass theorem still implies existence of a solution.

b) v_s, hence y_s, can take a continuum of value. There is an uncountably infinite number of possible histories. We are no longer dealing with elements of R^h. What do continuity and compactness mean in such a space? We need alternative sufficient conditions for a max to exist.

2. Is solving SP the easiest approach?

We need a preliminary result.

Law of iterated expectations: Given a random variable z and two nested informations sets $\Omega \subseteq H$ it is

$$\mathbb{E}(\mathbb{E}[z|H]|\Omega) = \mathbb{E}[z|\Omega].$$

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Law of iterated expectations: Given a random variable z and two nested informations sets $\Omega \subseteq H$ it is

$$\mathbb{E}(\mathbb{E}[z|H]|\Omega) = \mathbb{E}[z|\Omega].$$

e.g. suppose we have a stochastic process z_t and be I_t the information set available at time t. The law of iterated expectations implies $\mathbb{E}(\mathbb{E}[z_{t+h+s}|I_{t+h}]|I_t) = \mathbb{E}[z_{t+h+s}|I_t]$.

2. Is solving SP the easiest method? (II)

The law of iterated expectations allows to rewrite the SP problem as

$$W_s(a_s, z_s) = \tag{6}$$

$$\max_{\{c_s, a_{s+1}\}} \left[u(c_t) + \beta \mathbb{E}_t \max_{\{c_s, a_{s+1}\}_{s=t+1}^{T}} \sum_{s=t+1}^{T} \beta^{s-(t+1)} \mathbb{E}_{s+1} u\left(c_s\right) \right]$$

$$\text{s.t. } a_{s+1} = (1+r)\, a_s + y_s - c_s, \tag{7}$$

$$a_t \text{ given, } a_{T+1} \geq b \tag{8}$$

2. Is solving SP the easiest method? (III)

By definition of the value function $W_t(a_t, z_t)$ the above system can be rewritten in the

Dynamic programming or recursive problem (RP) form

$$W_s(a_s, z_s) = \max_{\{c_s, a_{s+1}\}} u(c_s) + \beta \mathbb{E}_s W_{s+1}(a_{s+1}, z_{s+1}) \tag{9}$$

$$\text{s.t. } a_{s+1} = (1+r)\, a_s + y_s - c_s, \tag{10}$$

$$a_t \text{ given, } a_{T+1} \geq b \tag{11}$$

Solving RP

For given T, b, a **solution** for RP is a triplet of *functions* $\{c_s(a_s, z_s), a_{s+1}(a_s, z_s), W_s(a_s, z_s)\}$ of time, beginning of period assets and current income realization such that

$$u'(c_s) = \beta \mathbb{E}_s \frac{\partial W_{s+1}(a_{s+1}, z_{s+1})}{\partial a_{s+1}}, \tag{12}$$

the Bellman equation (9), the dynamic constraint (10) plus the terminal condition hold (note that we are assuming W differentiable with respect to wealth).

If the function $W_{s+1}(a_{s+1}, z_{s+1})$ is known, (12) and (10) allow to solve for the functions $c_s(a_s, z_s), a_{s+1}(a_s, z_s)$.
$W_T(a_T, z_T) = u(c_T)$ is known and one can solve by working backwards.

Are SP and RP equivalent?

Two sets of questions:

1. When are SP and RP equivalent?
2. If they are which one is easier to use?

Are SP and RP equivalent?

The connection between the two problems, is Bellman's *Principle of Optimality.*

- If the value function for SP exists than a solution for SP generates functions $W_s(a_s, z_s), c_s(a_s, z_s)$ which together with (10) satisfy the Bellman equation (9); i.e. SP $\to$ RP. We have seen that Weierstrass theorem ensures SP has a solution if the solution is a vector in $\mathbb{R}^h$.
- If RP has a solution which satisfies some regularity condition then the path generated by the policy functions solve SP; RP $\to$ SP.

Are SP and RP equivalent? (II)

In the kind of problems we will consider the two problems are equivalent and we can solve whichever we find easier. In the presence of uncertainty it will be RP.

When is it easier to solve RP

- Functions are more complicated objects that vectors in $\mathbb{R}^h$, but...
- In the deterministic case there is no real advantage if we are interested in the solution for just one value of a_t, b. But if we are interested in the solution for various a_t, b we need to solve SP many times. The solution to RP holds for any a_t, b (it is a function).
- In the stochastic case, we need to work out the solution to SP *for any possible history!* The solution to RP applies for any realization of z_s.

When is it easier to solve RP? (II)

So, in many case RP is actually easier to solve, particularly for numerical problems which need to be solved with computers. In fact, dynamic programming is the tool of choice of numerical macro.

When is it easier to solve RP

If the horizon is infinite, $T = \infty$, both SP and RP look the same for all t in which a_t, z_t are the same. The value function and optimal sequences $\{c_s, a_{s+1}\}_{s=t}^{\infty}$ in SP are independent of t. Therefore we have

$$
\begin{aligned}
W(a_t, z_t) = & \max_{\{c_s, a_{s+1}\}_{s=t}^{\infty}} \sum_{s=t}^{\infty} \beta^{s-t} \mathbb{E}_t u(c_s) \\
\text{s.t. } & a_{s+1} = (1+r) a_s + y_s - c_s, \\
& a_t \text{ given.}
\end{aligned}
\tag{SP}
$$

and

$$
\begin{aligned}
W(a_s, z_s) = & \max_{\{c_s, a_{s+1}\}} u(c_s) + \beta \mathbb{E}_s W(a_{s+1}, z_{s+1}) \\
\text{s.t. } & a_{s+1} = (1+r) a_s + y_s - c_s, \\
& a_t \text{ given.}
\end{aligned}
\tag{RP}
$$

Infinite horizon

- Infinite horizon models are sometimes easier to solve (because of stationarity), sometimes more relevant (e.g. if agents are altruistic the economy may behave as if agents live forever).
- Yet, they involve additional complications. The solution to SP are now *infinite sequences* rather than vectors in $\mathbb{R}^h$. We cannot invoke Weierstrass theorem to guarantee existence of solution to SP!

Infinite horizon II

But think about what the conditions in Weierstrass theorem buy you.

1. Boundedness of the choice set and continuity of the objective function for any feasible plan buy you boundedness of the of the maximand $\rightarrow$ the sup of the maximand is bounded;
2. closeness of the choice set imply that the sup is internal to the set $\rightarrow$ the sup is a max.

Infinite horizon III

Existence of solution to SP with $T + \infty$ requires similar conditions:

1. Given z_s, a_s the set of values c_s, a_{s+1} satisfying the dynamic constraint must be compact.
2. The maximand must be bounded for all feasible plans.

Infinite horizon IV

Let us concentrate on 2.

- The maximand is an infinite sum. Bounded u is necessary but not sufficient for it to be bounded. We need $\beta < 1$.
- Usually u is assumed strictly increasing (unbounded). Some boundary conditions (as in the finite horizon case) must ensure that the maximand is bounded.
- One such boundary condition (but there can be many others) is the No-Ponzi-game condition you have already encountered in the Ramsey model. It requires $\lim_{s\to\infty} \frac{a_s}{(1+r)^s} \geq 0$.

Infinite horizon V

To see how it works, consider a deterministic model with $y_s = 0$ for all s. Consider a candidate bounded solution $\{c_s^*, a_{s+1}^*\}_{s=t}^{\infty}$ with $c_s < \bar{c}, \forall s$.

Consider the following alternative plan:

1. $\tilde{c}_t = c_t^* + 1$ which implies $a_{t+1} = a_{t+1}^* - 1$.
2. For all $s > t$ leave $\tilde{c}_s = c_s^*$ by borrowing to pay interests on the original extra loan of one; i.e. $\tilde{a}_{s+1} = a_{s+1}^* - (1+r)^{(s-t)}$.

With u strictly increasing this is an improvement on any candidate solution c_s^*. Therefore lifetime utility cannot achieve a max.

The No-Ponzi-game constraint rules out such a policy since it implies $\lim_{s\to\infty} \frac{a_s}{(1+r)^s} = -(1+r)^{-t} < 0$.

We will always need some kind of terminal condition to guarantee a solution to SP.

Method 1
Solving RP using the Bellman equation

Use the Bellman equation

$$W(a_s, z_s) = \max_{\{c_s, a_{s+1}\}} u(c_s) + \beta \mathbb{E}_s W(a_{s+1}, z_{s+1})$$

the optimality condition

$$u'(c_s) = \beta \mathbb{E}_s \frac{\partial W(a_{s+1}, z_{s+1})}{\partial a_{s+1}}$$

and the dynamics budget identity to solve *jointly* for $W(a, z)$, $c(a, z)$ and $a'(a, z)$.

We will use it just at the end of the course and talk further about it then.

Method 2:
Solving RP using the Euler equation

Suppose we know the value function $W(a_s, z_s)$ and that it is differentiable.
Using the dynamic constraint to replace for c_s in (RP) we can maximize with respect to a_{s+1} and obtain

$$u'(c_s) = \beta \mathbb{E}_s \frac{\partial W(a_{s+1}, z_{s+1})}{\partial a_{s+1}} \tag{13}$$

and given that u and W are known function we can solve for c.

Method 2:
Solving RP using the Euler equation II

But what if we do not know W? We can still do it!
Differentiate the value function $W(a_s, z_s)$ with respect to a_s to obtain

$$\frac{\partial W(a_s, z_s)}{\partial a_s} = u'(c_s)\frac{\partial c_s}{\partial a_s} + \beta \mathbb{E}_s \frac{\partial W(a_{s+1}, z_{s+1})}{\partial a_{s+1}} \left[(1+r) - \frac{\partial c_s}{\partial a_s}\right] \tag{14}$$

$$= \beta \mathbb{E}_s \frac{\partial W(a_{s+1}, z_{s+1})}{\partial a_{s+1}}(1+r) = (1+r)[u'(c_s)]. \tag{15}$$

Method 2: Solving RP using the Euler equation III

The above *envelope condition* allows to rewrite (13) as

$$u'(c_s) = \beta(1+r)\mathbb{E}_s u'(c_{s+1}). \tag{16}$$

The Euler equation does not contain the value function and one can use it to solve for the optimal policy $c(a, z)$ without needing to solve for W.

How do we go about doing this? Guess and verify.

Guess and verify the policy function

- Under appropriate regularity conditions the solution to the problem is unique.
- If a guess turns out to solve the maximization problem, then it is the unique solution.

Guess and verify the policy function: steps

1. Choose z_s. Trick: if y_s is of the form

$$y_s = g(v_{s-1}, v_{s-2}, ..., v_{s-k}) + u_s,$$

where v_s is a generic variable,* then $z_s = \{y_s, v_s, ..., v_{s-k+1}\}$ is a conservatively appropriate state variable.

2. Guess a functional form for $c(a_s, z_s)$ with unknown coefficients. This is more an art than a science, but most of our problems will have linear policy functions; i.e. $c_s = \alpha_0 + \alpha_1 a_s + \alpha_2 y_s + \alpha_3 v_s + \cdots + \alpha_{k+1} v_{s-k+1}$.

* Note that v_s here denotes a *generic* variable.

Guess and verify the policy function: steps

3. Use the guess for $c(a_s, z_s)$ to replace in the Euler equation and the dynamic budget constraint. Use one equation to replace appropriately into the other one. Impose that coefficients on each variable must add up to zero (otherwise the solution would not hold for any possible value of such variables).