Maths handout

1. Be $|q| < 1$. Then we have

$$\sum_{s=0}^{\infty} sq^s = \frac{q}{(1-q)^2} \tag{1}$$

Proof: We have

$$\sum_{s=0}^{\infty} sq^s = \left(q + 2q^2 + 3q^3 ...\right) = \tag{2}$$

$$q\left(1 + 2q + 3q^2\right) = q\left(1 + q + q + q^2 + q^2 + q^2 ...\right) = \tag{3}$$

$$q\left[1 + q + q^2 ... + q\left(1 + q + q^2 ...\right) + q^2\left(1 + q + q^2 ...\right)\right] = \tag{4}$$

$$q\left(1 + q + q^2 ...\right)\left(1 + q + q^2 ...\right) = q\frac{1}{1-q}\frac{1}{1-q} = \frac{q}{(1-q)^2}, \tag{5}$$

where the last line follows from the fact that $(1 + q + q^2 ...)$ is a geometric series and converges to $1/(1-q)$.

2. Be x a random variable with normal distribution $N(\mu, \sigma)$. Then γx has a normal distribution $N(\gamma\mu, |\gamma|\sigma)$.

3. Be x a random variable with normal distribution $N(\mu, \sigma)$. Then e^x has mean

$$E\left(e^x\right) = e^{\mu + \frac{\sigma^2}{2}}. \tag{6}$$

Proof:

$$E\left(e^{x}\right)=\int_{-\infty}^{+\infty}e^{x}\frac{e^{-\frac{(x-\mu)^{2}}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx= \tag{7}$$
$$\int_{-\infty}^{+\infty}\frac{e^{\frac{-x^{2}-\mu^{2}+2\mu x+2\sigma^{2}x}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx=\int_{-\infty}^{+\infty}\frac{e^{\frac{-x^{2}-\mu^{2}+2\mu x+2\sigma^{2}x-\sigma^{4}-2\sigma^{2}\mu+\sigma^{4}+2\sigma^{2}\mu}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx=$$
$$\int_{-\infty}^{+\infty}e^{\frac{\sigma^{4}+2\sigma^{2}\mu}{2\sigma^{2}}}\frac{e^{\frac{-x^{2}-\mu^{2}+2\mu x+2\sigma^{2}x-\sigma^{4}-2\sigma^{2}\mu}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx=$$
$$e^{\frac{\sigma^{2}}{2}+\mu}\int_{-\infty}^{+\infty}\frac{e^{\frac{-x^{2}-\mu^{2}+2\mu x+2\sigma^{2}x-\sigma^{4}-2\sigma^{2}\mu}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx=e^{\frac{\sigma^{2}}{2}+\mu}\int_{-\infty}^{+\infty}\frac{e^{\frac{-(x-\mu-\sigma^{2})^{2}}{2\sigma^{2}}}}{\sqrt{2\pi}\sigma}dx=e^{\frac{\sigma^{2}}{2}+\mu},$$

where the last equality follows from the fact that the integral of a normal density equals one.