Introduction

Chapter 1
The Nature of Derivatives

A derivative is an instrument whose value depends on the values of other more basic underlying variables.
Examples of Derivatives

• Futures Contracts
• Forward Contracts
• Swaps
• Options
Derivatives Markets

- Exchange traded
  - Traditionally exchanges have used the open-outcry system, but increasingly they are switching to electronic trading
  - Contracts are standard there is virtually no credit risk

- Over-the-counter (OTC)
  - A computer- and telephone-linked network of dealers at financial institutions, corporations, and fund managers
  - Contracts can be non-standard and there is some small amount of credit risk
Size of OTC and Exchange Markets
(Figure 1.1, Page 3)

Source: Bank for International Settlements. Chart shows total principal amounts for OTC market and value of underlying assets for exchange market.

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Ways Derivatives are Used

- To hedge risks
- To speculate (take a view on the future direction of the market)
- To lock in an arbitrage profit
- To change the nature of a liability
- To change the nature of an investment without incurring the costs of selling one portfolio and buying another
Forward Contracts

- Forward contracts are similar to futures except that they trade in the over-the-counter market.
- Forward contracts are particularly popular on currencies and interest rates.
### Foreign Exchange Quotes for GBP

**June 3, 2003** *(See page 4)*

<table>
<thead>
<tr>
<th></th>
<th>Bid</th>
<th>Offer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Spot</strong></td>
<td>1.6281</td>
<td>1.6285</td>
</tr>
<tr>
<td><strong>1-month forward</strong></td>
<td>1.6248</td>
<td>1.6253</td>
</tr>
<tr>
<td><strong>3-month forward</strong></td>
<td>1.6187</td>
<td>1.6192</td>
</tr>
<tr>
<td><strong>6-month forward</strong></td>
<td>1.6094</td>
<td>1.6100</td>
</tr>
</tbody>
</table>
Forward Price

- The forward price for a contract is the delivery price that would be applicable to the contract if were negotiated today (i.e., it is the delivery price that would make the contract worth exactly zero).

- The forward price may be different for contracts of different maturities.
Terminology

- The party that has agreed to buy has what is termed a long position
- The party that has agreed to sell has what is termed a short position
On June 3, 2003 the treasurer of a corporation enters into a long forward contract to buy £1 million in six months at an exchange rate of 1.6100

This obligates the corporation to pay $1,610,000 for £1 million on December 3, 2003

What are the possible outcomes?
Profit from a Long Forward Position

Profit

Price of Underlying at Maturity, $S_T$

$K$
Profit from a Short Forward Position

![Graph showing the profit from a short forward position as a function of the price of the underlying at maturity. The graph has a downward sloping line with the profit on the y-axis and the price of the underlying at maturity, $S_T$, on the x-axis. The line intersects the x-axis at $K$, indicating the break-even point.]
Futures Contracts (page 6)

- Agreement to buy or sell an asset for a certain price at a certain time
- Similar to forward contract
- Whereas a forward contract is traded OTC, a futures contract is traded on an exchange
Exchanges Trading Futures

- Chicago Board of Trade
- Chicago Mercantile Exchange
- LIFFE (London)
- Eurex (Europe)
- BM&F (Sao Paulo, Brazil)
- TIFFE (Tokyo)
- and many more (see list at end of book)
Examples of Futures Contracts

Agreement to:

- buy 100 oz. of gold @ US$400/oz. in December (NYMEX)
- sell £62,500 @ 1.5000 US$/£ in March (CME)
- sell 1,000 bbl. of oil @ US$20/bbl. in April (NYMEX)
1. Gold: An Arbitrage Opportunity?

Suppose that:

- The spot price of gold is US$300
- The 1-year forward price of gold is US$340
- The 1-year US$ interest rate is 5% per annum

Is there an arbitrage opportunity?
2. Gold: Another Arbitrage Opportunity?

Suppose that:
- The spot price of gold is US$300
- The 1-year forward price of gold is US$300
- The 1-year US$ interest rate is 5% per annum

Is there an arbitrage opportunity?
The Forward Price of Gold

If the spot price of gold is $S$ and the forward price for a contract deliverable in $T$ years is $F$, then

$$F = S (1+r)^T$$

where $r$ is the 1-year (domestic currency) risk-free rate of interest.

In our examples, $S = 300$, $T = 1$, and $r = 0.05$ so that

$$F = 300(1+0.05) = 315$$
1. Oil: An Arbitrage Opportunity?

Suppose that:
- The spot price of oil is US$19
- The quoted 1-year futures price of oil is US$25
- The 1-year US$ interest rate is 5% per annum
- The storage costs of oil are 2% per annum

Is there an arbitrage opportunity?
2. Oil: Another Arbitrage Opportunity?

Suppose that:

- The spot price of oil is US$19
- The quoted 1-year futures price of oil is US$16
- The 1-year US$ interest rate is 5% per annum
- The storage costs of oil are 2% per annum

Is there an arbitrage opportunity?
Options

- A call option is an option to buy a certain asset by a certain date for a certain price (the strike price)
- A put option is an option to sell a certain asset by a certain date for a certain price (the strike price)
American vs European Options

- An American option can be exercised at any time during its life.
- A European option can be exercised only at maturity.
Intel Option Prices (May 29, 2003; Stock Price=20.83); See Table 1.2 page 7

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>June Call</th>
<th>July Call</th>
<th>Oct Call</th>
<th>June Put</th>
<th>July Put</th>
<th>Oct Put</th>
</tr>
</thead>
<tbody>
<tr>
<td>20.00</td>
<td>1.25</td>
<td>1.60</td>
<td>2.40</td>
<td>0.45</td>
<td>0.85</td>
<td>1.50</td>
</tr>
<tr>
<td>22.50</td>
<td>0.20</td>
<td>0.45</td>
<td>1.15</td>
<td>1.85</td>
<td>2.20</td>
<td>2.85</td>
</tr>
</tbody>
</table>
Exchanges Trading Options

- Chicago Board Options Exchange
- American Stock Exchange
- Philadelphia Stock Exchange
- Pacific Exchange
- LIFFE (London)
- Eurex (Europe)
- and many more (see list at end of book)
Options vs Futures/Forwards

- A futures/forward contract gives the holder the obligation to buy or sell at a certain price
- An option gives the holder the right to buy or sell at a certain price
Types of Traders

- Hedgers
- Speculators
- Arbitrageurs

Some of the largest trading losses in derivatives have occurred because individuals who had a mandate to be hedgers or arbitrageurs switched to being speculators (See for example Barings Bank, Business Snapshot 1.2, page 15)
Hedging Examples (pages 10-11)

- A US company will pay £10 million for imports from Britain in 3 months and decides to hedge using a long position in a forward contract.

- An investor owns 1,000 Microsoft shares currently worth $28 per share. A two-month put with a strike price of $27.50 costs $1. The investor decides to hedge by buying 10 contracts.
Value of Microsoft Shares with and without Hedging (Fig 1.4, page 11)
Speculation Example

- An investor with $4,000 to invest feels that Amazon.com’s stock price will increase over the next 2 months. The current stock price is $40 and the price of a 2-month call option with a strike of 45 is $2
- What are the alternative strategies?
Arbitrage Example

- A stock price is quoted as £100 in London and $172 in New York
- The current exchange rate is 1.7500
- What is the arbitrage opportunity?
Hedge Funds  (see Business Snapshot 1.1, page 9)

- Hedge funds are not subject to the same rules as mutual funds and cannot offer their securities publicly.
- Mutual funds must
  - disclose investment policies,
  - makes shares redeemable at any time,
  - limit use of leverage
  - take no short positions.
- Hedge funds are not subject to these constraints.
- Hedge funds use complex trading strategies: big users of derivatives for hedging, speculation and arbitrage
Futures Contracts

- Available on a wide range of underlyings
- Exchange traded
- Specifications need to be defined:
  - What can be delivered,
  - Where it can be delivered, &
  - When it can be delivered
- Settled daily
Margins

- A margin is cash or marketable securities deposited by an investor with his or her broker.
- The balance in the margin account is adjusted to reflect daily settlement.
- Margins minimize the possibility of a loss through a default on a contract.
Example of a Futures Trade (page 27-28)

- An investor takes a long position in 2 December gold futures contracts on June 5
  - contract size is 100 oz.
  - futures price is US$400
  - margin requirement is US$2,000/contract (US$4,000 in total)
  - maintenance margin is US$1,500/contract (US$3,000 in total)
### A Possible Outcome

#### Table 2.1, Page 28

<table>
<thead>
<tr>
<th>Day</th>
<th>Futures Price (US$)</th>
<th>Daily Gain (Loss) (US$)</th>
<th>Cumulative Gain (Loss) (US$)</th>
<th>Account Balance (US$)</th>
<th>Margin Call (US$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>400.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5-Jun</td>
<td>397.00</td>
<td>(600)</td>
<td>(600)</td>
<td>3,400</td>
<td>0</td>
</tr>
<tr>
<td>13-Jun</td>
<td>393.30</td>
<td>(420)</td>
<td>(1,340)</td>
<td>2,660</td>
<td>+ 1,340 = 4,000</td>
</tr>
<tr>
<td>19-Jun</td>
<td>387.00</td>
<td>(1,140)</td>
<td>(2,600)</td>
<td>2,740</td>
<td>+ 1,260 = 4,000</td>
</tr>
<tr>
<td>26-Jun</td>
<td>392.30</td>
<td>260</td>
<td>(1,540)</td>
<td>5,060</td>
<td>0</td>
</tr>
</tbody>
</table>

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Other Key Points About Futures

- They are settled daily
- Closing out a futures position involves entering into an offsetting trade
- Most contracts are closed out before maturity
Collateralization in OTC Markets

- It is becoming increasingly common for contracts to be collateralized in OTC markets.
- They are then similar to futures contracts in that they are settled regularly (e.g. every day or every week).
Futures Prices for Gold on Feb 4, 2004: Prices Increase with Maturity (Figure 2.2, page 35)

(a) Gold

Contract Maturity Month

Futures Price ($ per oz)
Futures Prices for Oil on February 4, 2004: Prices Decrease with Maturity (Figure 2.2, page 35)
Delivery

- If a futures contract is not closed out before maturity, it is usually settled by delivering the assets underlying the contract. When there are alternatives about what is delivered, where it is delivered, and when it is delivered, the party with the short position chooses.

- A few contracts (for example, those on stock indices and Eurodollars) are settled in cash
Some Terminology

- **Open interest**: the total number of contracts outstanding
  - equal to number of long positions or number of short positions
- **Settlement price**: the price just before the final bell each day
  - used for the daily settlement process
- **Volume of trading**: the number of trades in 1 day
Convergence of Futures to Spot

(Figure 2.1, page 26)
Questions

- When a new trade is completed what are the possible effects on the open interest?
- Can the volume of trading in a day be greater than the open interest?
Regulation of Futures

- Regulation is designed to protect the public interest.
- Regulators try to prevent questionable trading practices by either individuals on the floor of the exchange or outside groups.
Accounting & Tax

- Ideally hedging profits (losses) should be recognized at the same time as the losses (profits) on the item being hedged.
- Ideally profits and losses from speculation should be recognized on a mark-to-market basis.
- Roughly speaking, this is what the accounting and tax treatment of futures in the U.S. and many other countries attempts to achieve.
# Forward Contracts vs Futures Contracts

<table>
<thead>
<tr>
<th></th>
<th>FORWARDS</th>
<th>FUTURES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Private contract between 2 parties</td>
<td>Exchange traded</td>
<td></td>
</tr>
<tr>
<td>Non-standard contract</td>
<td>Standard contract</td>
<td></td>
</tr>
<tr>
<td>Usually 1 specified delivery date</td>
<td>Range of delivery dates</td>
<td></td>
</tr>
<tr>
<td>Settled at end of contract</td>
<td>Settled daily</td>
<td></td>
</tr>
<tr>
<td>Delivery or final cash settlement usually occurs</td>
<td>Contract usually closed out prior to maturity</td>
<td>Virtually no credit risk</td>
</tr>
<tr>
<td>Some credit risk</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Some credit risk</td>
</tr>
</tbody>
</table>
Foreign Exchange Quotes

- Futures exchange rates are quoted as the number of USD per unit of the foreign currency.
- Forward exchange rates are quoted in the same way as spot exchange rates. This means that GBP, EUR, AUD, and NZD are quoted as USD per unit of foreign currency. Other currencies (e.g., CAD and JPY) are quoted as units of the foreign currency per USD.
Hedging Strategies Using Futures

Chapter 3
Long & Short Hedges

- A long futures hedge is appropriate when you know you will purchase an asset in the future and want to lock in the price.
- A short futures hedge is appropriate when you know you will sell an asset in the future & want to lock in the price.
Arguments in Favor of Hedging

Companies should focus on the main business they are in and take steps to minimize risks arising from interest rates, exchange rates, and other market variables.
Arguments against Hedging

- Shareholders are usually well diversified and can make their own hedging decisions.
- It may increase risk to hedge when competitors do not.
- Explaining a situation where there is a loss on the hedge and a gain on the underlying can be difficult.
Convergence of Futures to Spot
(Hedge initiated at time $t_1$ and closed out at time $t_2$)
Basis Risk

- Basis is the difference between spot & futures
- Basis risk arises because of the uncertainty about the basis when the hedge is closed out
Long Hedge

- Suppose that
  
  $F_1$ : Initial Futures Price  
  $F_2$ : Final Futures Price  
  $S_2$ : Final Asset Price  

- You hedge the future purchase of an asset by entering into a long futures contract

- Cost of Asset $= S_2 - (F_2 - F_1) = F_1 + \text{Basis}$
Short Hedge

- Suppose that
  \[ F_1 : \text{Initial Futures Price} \]
  \[ F_2 : \text{Final Futures Price} \]
  \[ S_2 : \text{Final Asset Price} \]
- You hedge the future sale of an asset by entering into a short futures contract
- Price Realized = \[ S_2 + (F_1 - F_2) = F_1 + \text{Basis} \]
Choice of Contract

- Choose a delivery month that is as close as possible to, but later than, the end of the life of the hedge
- When there is no futures contract on the asset being hedged, choose the contract whose futures price is most highly correlated with the asset price. This is known as cross hedging.
Optimal Hedge Ratio

Proportion of the exposure that should optimally be hedged is

\[ \rho \frac{\sigma_S}{\sigma_F} \]

where

\( \sigma_S \) is the standard deviation of \( \Delta S \), the change in the spot price during the hedging period,
\( \sigma_F \) is the standard deviation of \( \Delta F \), the change in the futures price during the hedging period
\( \rho \) is the coefficient of correlation between \( \Delta S \) and \( \Delta F \).
Hedging Using Index Futures
(Page 63)

To hedge the risk in a portfolio the number of contracts that should be shorted is

$$\beta \frac{P}{A}$$

where $P$ is the value of the portfolio, $\beta$ is its beta, and $A$ is the value of the assets underlying one futures contract
Reasons for Hedging an Equity Portfolio

- Desire to be out of the market for a short period of time. (Hedging may be cheaper than selling the portfolio and buying it back.)
- Desire to hedge systematic risk (Appropriate when you feel that you have picked stocks that will outperform the market.)
Example

Value of S&P 500 is 1,000
Value of Portfolio is $5 million
Beta of portfolio is 1.5

What position in futures contracts on the S&P 500 is necessary to hedge the portfolio?
Changing Beta

- What position is necessary to reduce the beta of the portfolio to 0.75?
- What position is necessary to increase the beta of the portfolio to 2.0?
Hedging Price of an Individual Stock

- Similar to hedging a portfolio
- Does not work as well because only the systematic risk is hedged
- The unsystematic risk that is unique to the stock is not hedged
Why Hedge Equity Returns

- May want to be out of the market for a while. Hedging avoids the costs of selling and repurchasing the portfolio.
- Suppose stocks in your portfolio have an average beta of 1.0, but you feel they have been chosen well and will outperform the market in both good and bad times. Hedging ensures that the return you earn is the risk-free return plus the excess return of your portfolio over the market.
We can use a series of futures contracts to increase the life of a hedge.

Each time we switch from 1 futures contract to another we incur a type of basis risk.
Types of Rates

- Treasury rates
- LIBOR rates
- Repo rates
Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement.
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers.
Continuous Compounding

(Page 79)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- $100$ grows to $100e^{RT}$ when invested at a continuously compounded rate $R$ for time $T$
- $100$ received at time $T$ discounts to $100e^{-RT}$ at time zero when the continuously compounded discount rate is $R$
Define

\( R_c \) : continuously compounded rate
\( R_m \) : same rate with compounding \( m \) times per year

\[
R_c = m \ln \left( 1 + \frac{R_m}{m} \right)
\]

\[
R_m = m \left( e^{R_c/m} - 1 \right)
\]
Zero Rates

A zero rate (or spot rate), for maturity $T$ is the rate of interest earned on an investment that provides a payoff only at time $T$. 
Example  (Table 4.2, page 81)

<table>
<thead>
<tr>
<th>Maturity (years)</th>
<th>Zero Rate (% cont comp)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5.0</td>
</tr>
<tr>
<td>1.0</td>
<td>5.8</td>
</tr>
<tr>
<td>1.5</td>
<td>6.4</td>
</tr>
<tr>
<td>2.0</td>
<td>6.8</td>
</tr>
</tbody>
</table>
Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate.

- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

\[
3e^{-0.05\times0.5} + 3e^{-0.058\times1.0} + 3e^{-0.064\times1.5} \\
+ 103e^{-0.068\times2.0} = 98.39
\]
Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond.

- Suppose that the market price of the bond in our example equals its theoretical price of 98.39.

- The bond yield (continuously compounded) is given by solving

\[ 3e^{-y\times0.5} + 3e^{-y\times1.0} + 3e^{-y\times1.5} + 103e^{-y\times2.0} = 98.39 \]

to get \( y = 0.0676 \) or 6.76%.
Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
- In our example we solve

\[
\frac{c}{2} e^{-0.05 \times 0.5} + \frac{c}{2} e^{-0.058 \times 1.0} + \frac{c}{2} e^{-0.064 \times 1.5} \\
+ \left( 100 + \frac{c}{2} \right) e^{-0.068 \times 2.0} = 100
\]

to get \( c = 6.87 \) (with s.a. compounding)
Par Yield continued

In general if \( m \) is the number of coupon payments per year, \( P \) is the present value of $1 received at maturity and \( A \) is the present value of an annuity of $1 on each coupon date

\[
C = \frac{(100 - 100P)m}{A}
\]
## Sample Data (Table 4.3, page 82)

<table>
<thead>
<tr>
<th>Bond Principal (dollars)</th>
<th>Time to Maturity (years)</th>
<th>Annual Coupon (dollars)</th>
<th>Bond Cash Price (dollars)</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>0.25</td>
<td>0</td>
<td>97.5</td>
</tr>
<tr>
<td>100</td>
<td>0.50</td>
<td>0</td>
<td>94.9</td>
</tr>
<tr>
<td>100</td>
<td>1.00</td>
<td>0</td>
<td>90.0</td>
</tr>
<tr>
<td>100</td>
<td>1.50</td>
<td>8</td>
<td>96.0</td>
</tr>
<tr>
<td>100</td>
<td>2.00</td>
<td>12</td>
<td>101.6</td>
</tr>
</tbody>
</table>
The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- The 3-month rate is 4 times 2.5/97.5 or 10.256% with quarterly compounding.
- This is 10.127% with continuous compounding.
- Similarly, the 6-month and 1-year rates are 10.469% and 10.536% with continuous compounding.
The Bootstrap Method continued

- To calculate the 1.5 year rate we solve

\[ 4e^{-0.10469 \times 0.5} + 4e^{-0.10536 \times 1.0} + 104e^{-R \times 1.5} = 96 \]

\[ R = 0.10681 \] or 10.681%

- Similarly the two-year rate is 10.808%
Zero Curve Calculated from the Data (Figure 4.1, page 84)
Forward Rates

The forward rate is the future zero rate implied by today’s term structure of interest rates
### Calculation of Forward Rates

Table 4.5, page 85

<table>
<thead>
<tr>
<th>Year ((n) )</th>
<th>Zero Rate for an (n)-year Investment (% per annum)</th>
<th>Forward Rate for (n)th Year (% per annum)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.0</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4.0</td>
<td>5.0</td>
</tr>
<tr>
<td>3</td>
<td>4.6</td>
<td>5.8</td>
</tr>
<tr>
<td>4</td>
<td>5.0</td>
<td>6.2</td>
</tr>
<tr>
<td>5</td>
<td>5.3</td>
<td>6.5</td>
</tr>
</tbody>
</table>
Formula for Forward Rates

- Suppose that the zero rates for time periods $T_1$ and $T_2$ are $R_1$ and $R_2$ with both rates continuously compounded.
- The forward rate for the period between times $T_1$ and $T_2$ is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$
Instantaneous Forward Rate

- The instantaneous forward rate for a maturity $T$ is the forward rate that applies for a very short time period starting at $T$. It is

$$R + T \frac{\partial R}{\partial T}$$

where $R$ is the $T$-year rate.
Upward vs Downward Sloping Yield Curve

- For an upward sloping yield curve: 
  Fwd Rate > Zero Rate > Par Yield

- For a downward sloping yield curve 
  Par Yield > Zero Rate > Fwd Rate
Forward Rate Agreement

- A forward rate agreement (FRA) is an agreement that a certain rate will apply to a certain principal during a certain future time period.
Forward Rate Agreement continued

- An FRA is equivalent to an agreement where interest at a predetermined rate, $R_K$, is exchanged for interest at the market rate.
- An FRA can be valued by assuming that the forward interest rate is certain to be realized.
Valuation Formulas (equations 4.9 and 4.10 page 88)

- Value of FRA where a fixed rate $R_K$ will be received on a principal $L$ between times $T_1$ and $T_2$ is:
  \[ L(R_K - R_F)(T_2 - T_1)e^{-R_2 T_2} \]

- Value of FRA where a fixed rate is paid is:
  \[ L(R_F - R_K)(T_2 - T_1)e^{-R_2 T_2} \]

- $R_F$ is the forward rate for the period and $R_2$ is the zero rate for maturity $T_2$

- What compounding frequencies are used in these formulas for $R_K$, $R_M$, and $R_2$?
Duration (page 89)

- Duration of a bond that provides cash flow $c_i$ at time $t_i$ is
  \[
  \sum_{i=1}^{n} t_i \left[ \frac{c_i e^{-yt_i}}{B} \right]
  \]
  where $B$ is its price and $y$ is its yield (continuously compounded).

- This leads to
  \[
  \frac{\Delta B}{B} = -D \Delta y
  \]
Duration Continued

- When the yield $y$ is expressed with compounding $m$ times per year

$$\Delta B = -\frac{BD\Delta y}{1 + y/m}$$

- The expression

$$\frac{D}{1 + y/m}$$

is referred to as the “modified duration”
Convexity

The convexity of a bond is defined as

\[
C = \frac{1}{B} \frac{\partial^2 B}{\partial y^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-y t_i}}{B}
\]

so that

\[
\frac{\Delta B}{B} = -D \Delta y + \frac{1}{2} C (\Delta y)^2
\]
Theories of the Term Structure
Page 93

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates
Determination of Forward and Futures Prices

Chapter 5
Consumption vs Investment Assets

- Investment assets are assets held by significant numbers of people purely for investment purposes (Examples: gold, silver)
- Consumption assets are assets held primarily for consumption (Examples: copper, oil)
Short selling involves selling securities you do not own.

Your broker borrows the securities from another client and sells them in the market in the usual way.
Short Selling
(continued)

- At some stage you must buy the securities back so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives
Notation for Valuing Futures and Forward Contracts

\[ S_0: \text{Spot price today} \]
\[ F_0: \text{Futures or forward price today} \]
\[ T: \text{Time until delivery date} \]
\[ r: \text{Risk-free interest rate for maturity } T \]
1. Gold: An Arbitrage Opportunity?

- Suppose that:
  - The spot price of gold is US$390
  - The quoted 1-year forward price of gold is US$425
  - The 1-year US$ interest rate is 5% per annum
  - No income or storage costs for gold
- Is there an arbitrage opportunity?
2. Gold: Another Arbitrage Opportunity?

- Suppose that:
  - The spot price of gold is US$390
  - The quoted 1-year forward price of gold is US$390
  - The 1-year US$ interest rate is 5% per annum
  - No income or storage costs for gold
- Is there an arbitrage opportunity?
The Forward Price of Gold

If the spot price of gold is $S$ and the futures price is $F$ for a contract deliverable in $T$ years is $F$, then

$$F = S (1+r)^T$$

where $r$ is the 1-year (domestic currency) risk-free rate of interest.

In our examples, $S=390$, $T=1$, and $r=0.05$ so that

$$F = 390(1+0.05) = 409.50$$
When Interest Rates are Measured with Continuous Compounding

\[ F_0 = S_0 e^{rT} \]

This equation relates the forward price and the spot price for any investment asset that provides no income and has no storage costs.
When an Investment Asset Provides a Known Dollar Income
(page 105, equation 5.2)

\[ F_0 = (S_0 - I) e^{rT} \]

where \( I \) is the present value of the income during life of forward contract.
When an Investment Asset Provides a Known Yield
(Page 107, equation 5.3)

\[ F_0 = S_0 e^{(r-q)T} \]

where \( q \) is the average yield during the life of the contract (expressed with continuous compounding)
Valuing a Forward Contract

Page 108

- Suppose that
  - $K$ is delivery price in a forward contract and
  - $F_0$ is forward price that would apply to the contract today
- The value of a long forward contract, $f$, is
  \[ f = (F_0 - K) e^{-rT} \]
- Similarly, the value of a short forward contract is
  \[ (K - F_0) e^{-rT} \]
Forward vs Futures Prices

- Forward and futures prices are usually assumed to be the same. When interest rates are uncertain they are, in theory, slightly different:
  - A strong positive correlation between interest rates and the asset price implies the futures price is slightly higher than the forward price
  - A strong negative correlation implies the reverse
Stock Index (Page 110-112)

- Can be viewed as an investment asset paying a dividend yield
- The futures price and spot price relationship is therefore

\[ F_0 = S_0 e^{(r-q)T} \]

where \( q \) is the average dividend yield on the portfolio represented by the index during life of contract
Stock Index (continued)

- For the formula to be true it is important that the index represent an investment asset.
- In other words, changes in the index must correspond to changes in the value of a tradable portfolio.
- The Nikkei index viewed as a dollar number does not represent an investment asset (See Business Snapshot 5.3, page 111).
Index Arbitrage

- When \( F_0 > S_0 e^{(r-q)T} \) an arbitrageur buys the stocks underlying the index and sells futures
- When \( F_0 < S_0 e^{(r-q)T} \) an arbitrageur buys futures and shorts or sells the stocks underlying the index
Index Arbitrage
(continued)

- Index arbitrage involves simultaneous trades in futures and many different stocks.
- Very often a computer is used to generate the trades.
- Occasionally (e.g., on Black Monday) simultaneous trades are not possible and the theoretical no-arbitrage relationship between $F_0$ and $S_0$ does not hold.
A foreign currency is analogous to a security providing a dividend yield.

The continuous dividend yield is the foreign risk-free interest rate.

It follows that if $r_f$ is the foreign risk-free interest rate,

$$F_0 = S_0 e^{(r-r_f)T}$$
Why the Relation Must Be True

Figure 5.1, page 113

1000 units of foreign currency at time zero

$1000 e^{r_f T}$
units of foreign currency at time $T$

$1000F_0 e^{r_f T}$
dollars at time $T$

$1000S_0$ dollars at time zero

$1000S_0 e^{r T}$
dollars at time $T$
Futures on Consumption Assets
(Page 117-118)

\[ F_0 \leq S_0 e^{(r+u)T} \]

where \( u \) is the storage cost per unit time as a percent of the asset value.

Alternatively,

\[ F_0 \leq (S_0 + U)e^{rT} \]

where \( U \) is the present value of the storage costs.
The Cost of Carry (Page 118-119)

- The cost of carry, \( c \), is the storage cost plus the interest costs less the income earned
- For an investment asset: \( F_0 = S_0 e^{cT} \)
- For a consumption asset: \( F_0 \leq S_0 e^{cT} \)
- The convenience yield on the consumption asset, \( y \), is defined so that
  \[
  F_0 = S_0 e^{(c-y)T}
  \]
Suppose $k$ is the expected return required by investors on an asset.

We can invest $F_0e^{-rT}$ at the risk-free rate and enter into a long futures contract so that there is a cash inflow of $S_T$ at maturity.

This shows that

$$(F_0e^{-rT})e^{kT} = E(S_T)$$

or

$$F_0 = E(S_T)e^{(r-k)T}$$
Futures Prices & Future Spot Prices (continued)

- If the asset has
  - no systematic risk, then \( k = r \) and \( F_0 \) is an unbiased estimate of \( S_T \)
  - positive systematic risk, then \( k > r \) and \( F_0 < E(S_T) \)
  - negative systematic risk, then \( k < r \) and \( F_0 > E(S_T) \)
Interest Rate Futures

Chapter 6
Day Count Conventions in the U.S. (Page 129)

Treasury Bonds: Actual/Actual (in period)

Corporate Bonds: 30/360

Money Market Instruments: Actual/360
Treasury Bond Price Quotes in the U.S

Cash price = Quoted price + Accrued Interest
Cash price received by party with short position =
Most Recent Settlement Price ×
Conversion factor + Accrued interest
Example

- Settlement price of bond delivered = 90.00
- Conversion factor = 1.3800
- Accrued interest on bond = 3.00
- Price received for bond is
  \[1.3800 \times 9.00) + 3.00 = \$127.20\]
  per $100 of principal
Conversion Factor

The conversion factor for a bond is approximately equal to the value of the bond on the assumption that the yield curve is flat at 6% with semiannual compounding.
CBOT
T-Bonds & T-Notes

Factors that affect the futures price:

- Delivery can be made any time during the delivery month
- Any of a range of eligible bonds can be delivered
- The wild card play
Eurodollar Futures

- A Eurodollar is a dollar deposited in a bank outside the United States.
- Eurodollar futures are futures on the 3-month Eurodollar deposit rate (same as 3-month LIBOR rate).
- One contract is on the rate earned on $1 million.
- A change of one basis point or 0.01 in a Eurodollar futures quote corresponds to a contract price change of $25.
Eurodollar Futures continued

- A Eurodollar futures contract is settled in cash
- When it expires (on the third Wednesday of the delivery month) the final settlement price is 100 minus the actual three month deposit rate
Example

- Suppose you buy (take a long position in) a contract on November 1
- The contract expires on December 21
- The prices are as shown
- How much do you gain or lose a) on the first day, b) on the second day, c) over the whole time until expiration?
Example

<table>
<thead>
<tr>
<th>Date</th>
<th>Quote</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nov 1</td>
<td>97.12</td>
</tr>
<tr>
<td>Nov 2</td>
<td>97.23</td>
</tr>
<tr>
<td>Nov 3</td>
<td>96.98</td>
</tr>
<tr>
<td>.......</td>
<td>.......</td>
</tr>
<tr>
<td>Dec 21</td>
<td>97.42</td>
</tr>
</tbody>
</table>
Example continued

- If on Nov. 1 you know that you will have $1 million to invest on for three months on Dec 21, the contract locks in a rate of $100 - 97.12 = 2.88%.

- In the example you earn 100 – 97.42 = 2.58% on $1 million for three months (= $6,450) and make a gain day by day on the futures contract of 30 × $25 = $750.
If \( Q \) is the quoted price of a Eurodollar futures contract, the value of one contract is 10,000\([100-0.25(100-Q)]\)
Forward Rates and Eurodollar Futures (Page 139-142)

- Eurodollar futures contracts last as long as 10 years.
- For Eurodollar futures lasting beyond two years we cannot assume that the forward rate equals the futures rate.
There are Two Reasons

- Futures is settled daily where forward is settled once
- Futures is settled at the beginning of the underlying three-month period; forward is settled at the end of the underlying three-month period
Forward Rates and Eurodollar Futures continued

A "convexity adjustment" often made is

Forward rate = Futures rate − \( \frac{1}{2} \sigma^2 t_1 t_2 \)

where \( t_1 \) is the time to maturity of the futures contract, \( t_2 \) is the maturity of the rate underlying the futures contract (90 days later than \( t_1 \)) and \( \sigma \) is the standard deviation of the short rate changes per year (typically \( \sigma \) is about 0.012)
Convexity Adjustment when $\sigma = 0.012$ (Table 6.3, page 141)

<table>
<thead>
<tr>
<th>Maturity of Futures</th>
<th>Convexity Adjustment (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>3.2</td>
</tr>
<tr>
<td>4</td>
<td>12.2</td>
</tr>
<tr>
<td>6</td>
<td>27.0</td>
</tr>
<tr>
<td>8</td>
<td>47.5</td>
</tr>
<tr>
<td>10</td>
<td>73.8</td>
</tr>
</tbody>
</table>
Extending the LIBOR Zero Curve

- LIBOR deposit rates define the LIBOR zero curve out to one year
- Eurodollar futures can be used to determine forward rates and the forward rates can then be used to bootstrap the zero curve
Example

\[ F_i = \frac{R_{i+1} T_{i+1} - R_i T_i}{T_{i+1} - T_i} \]

so that

\[ R_{i+1} = \frac{F_i (T_{i+1} - T_i) + R_i T_i}{T_{i+1}} \]

If the 400 day LIBOR rate has been calculated as 4.80% and the forward rate for the period between 400 and 491 days is 5.30 the 491 days rate is 4.893%
Duration Matching

- This involves hedging against interest rate risk by matching the durations of assets and liabilities
- It provides protection against small parallel shifts in the zero curve
Use of Eurodollar Futures

- One contract locks in an interest rate on $1 million for a future 3-month period.
- How many contracts are necessary to lock in an interest rate for a future six month period?
Duration-Based Hedge Ratio

\[ \frac{PD_P}{F_CF_DF} \]

- **\(F_C\)**: Contract price for interest rate futures
- **\(D_F\)**: Duration of asset underlying futures at maturity
- **\(P\)**: Value of portfolio being hedged
- **\(D_P\)**: Duration of portfolio at hedge maturity
Example

- It is August. A fund manager has $10 million invested in a portfolio of government bonds with a duration of 6.80 years and wants to hedge against interest rate moves between August and December.
- The manager decides to use December T-bond futures. The futures price is 93-02 or 93.0625 and the duration of the cheapest to deliver bond is 9.2 years.
- The number of contracts that should be shorted is

\[
\frac{10,000,000}{93,062.50} \times \frac{6.80}{9.20} = 79
\]
Limitations of Duration-Based Hedging

- Assumes that only parallel shift in yield curve take place
- Assumes that yield curve changes are small
GAP Management  (Business Snapshot 6.3)

This is a more sophisticated approach used by banks to hedge interest rate. It involves

- Bucketing the zero curve
- Hedging exposure to situation where rates corresponding to one bucket change and all other rates stay the same.
A swap is an agreement to exchange cash flows at specified future times according to certain specified rules.
An Example of a “Plain Vanilla” Interest Rate Swap

- An agreement by Microsoft to receive 6-month LIBOR & pay a fixed rate of 5% per annum every 6 months for 3 years on a notional principal of $100 million
- Next slide illustrates cash flows
## Cash Flows to Microsoft
(See Table 7.1, page 151)

<table>
<thead>
<tr>
<th>Date</th>
<th>Rate</th>
<th>Floating Cash Flow</th>
<th>Fixed Cash Flow</th>
<th>Net Cash Flow</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mar. 5, 2004</td>
<td>4.2%</td>
<td>2.10</td>
<td>-2.50</td>
<td>-0.40</td>
</tr>
<tr>
<td>Sept. 5, 2004</td>
<td>4.8%</td>
<td>+2.10</td>
<td>-2.50</td>
<td>-0.40</td>
</tr>
<tr>
<td>Mar. 5, 2005</td>
<td>5.3%</td>
<td>+2.40</td>
<td>-2.50</td>
<td>-0.10</td>
</tr>
<tr>
<td>Sept. 5, 2005</td>
<td>5.5%</td>
<td>+2.65</td>
<td>-2.50</td>
<td>+0.15</td>
</tr>
<tr>
<td>Mar. 5, 2006</td>
<td>5.6%</td>
<td>+2.75</td>
<td>-2.50</td>
<td>+0.25</td>
</tr>
<tr>
<td>Sept. 5, 2006</td>
<td>5.9%</td>
<td>+2.80</td>
<td>-2.50</td>
<td>+0.30</td>
</tr>
<tr>
<td>Mar. 5, 2007</td>
<td>6.4%</td>
<td>+2.95</td>
<td>-2.50</td>
<td>+0.45</td>
</tr>
</tbody>
</table>

---------Millions of Dollars---------
Typical Uses of an Interest Rate Swap

- Converting a liability from
  - fixed rate to floating rate
  - floating rate to fixed rate

- Converting an investment from
  - fixed rate to floating rate
  - floating rate to fixed rate
Intel and Microsoft (MS) Transform a Liability
(Figure 7.2, page 152)
Financial Institution is Involved
(Figure 7.4, page 153)

Intel → LIBOR

LIBOR → 4.985%

F.I. → LIBOR

LIBOR → 5.015%

MS → LIBOR + 0.1%

5.2%
Intel and Microsoft (MS) Transform an Asset
(Figure 7.3, page 153)
Financial Institution is Involved
(See Figure 7.5, page 154)
Quotes By a Swap Market Maker
(Table 7.3, page 155)

<table>
<thead>
<tr>
<th>Maturity</th>
<th>Bid (%)</th>
<th>Offer (%)</th>
<th>Swap Rate (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 years</td>
<td>6.03</td>
<td>6.06</td>
<td>6.045</td>
</tr>
<tr>
<td>3 years</td>
<td>6.21</td>
<td>6.24</td>
<td>6.225</td>
</tr>
<tr>
<td>4 years</td>
<td>6.35</td>
<td>6.39</td>
<td>6.370</td>
</tr>
<tr>
<td>5 years</td>
<td>6.47</td>
<td>6.51</td>
<td>6.490</td>
</tr>
<tr>
<td>7 years</td>
<td>6.65</td>
<td>6.68</td>
<td>6.665</td>
</tr>
<tr>
<td>10 years</td>
<td>6.83</td>
<td>6.87</td>
<td>6.850</td>
</tr>
</tbody>
</table>
The Comparative Advantage Argument (Table 7.4, page 157)

- AAACorp wants to borrow floating
- BBBCorp wants to borrow fixed

<table>
<thead>
<tr>
<th></th>
<th>Fixed</th>
<th>Floating</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAACorp</td>
<td>4.0%</td>
<td>6-month LIBOR + 0.30%</td>
</tr>
<tr>
<td>BBBCorp</td>
<td>5.20%</td>
<td>6-month LIBOR + 1.00%</td>
</tr>
</tbody>
</table>
The Swap (Figure 7.6, page 158)

AAACorp \[\rightarrow\] LIBOR \[\leftarrow\] BBBCorp

4% \[\rightarrow\] 3.95% \[\leftarrow\] 4%

LIBOR \[\rightarrow\] LIBOR+1% \[\leftarrow\]
The Swap when a Financial Institution is Involved
(Figure 7.7, page 158)
Criticisms of the Comparative Advantage Argument

- The 4.0% and 5.2% rates available to AAACorp and BBBCorp in fixed rate markets are 5-year rates.
- The LIBOR+0.3% and LIBOR+1% rates available in the floating rate market are six-month rates.
- BBBCorp’s fixed rate depends on the spread above LIBOR it borrows at in the future.
The Nature of Swap Rates

- Six-month LIBOR is a short-term AA borrowing rate.
- The 5-year swap rate has a risk corresponding to the situation where 10 six-month loans are made to AA borrowers at LIBOR.
- This is because the lender can enter into a swap where income from the LIBOR loans is exchanged for the 5-year swap rate.
Using Swap Rates to Bootstrap the LIBOR/Swap Zero Curve

- Consider a new swap where the fixed rate is the swap rate.
- When principals are added to both sides on the final payment date the swap is the exchange of a fixed rate bond for a floating rate bond.
- The floating-rate rate bond is worth par. The swap is worth zero. The fixed-rate bond must therefore also be worth par.
- This shows that swap rates define par yield bonds that can be used to bootstrap the LIBOR (or LIBORswap) zero curve.
Valuation of an Interest Rate Swap that is not New

- Interest rate swaps can be valued as the difference between the value of a fixed-rate bond and the value of a floating-rate bond
- Alternatively, they can be valued as a portfolio of forward rate agreements (FRAs)
Valuation in Terms of Bonds

- The fixed rate bond is valued in the usual way.
- The floating rate bond is valued by noting that it is worth par immediately after the next payment date.
Valuation in Terms of FRAs

- Each exchange of payments in an interest rate swap is an FRA
- The FRAs can be valued on the assumption that today’s forward rates are realized
An Example of a Currency Swap

An agreement to pay 11% on a sterling principal of £10,000,000 & receive 8% on a US$ principal of $15,000,000 every year for 5 years
Exchange of Principal

- In an interest rate swap the principal is not exchanged.
- In a currency swap the principal is usually exchanged at the beginning and the end of the swap’s life.
### The Cash Flows (Table 7.7, page 166)

<table>
<thead>
<tr>
<th>Year</th>
<th>Dollars $</th>
<th>Pounds £</th>
</tr>
</thead>
<tbody>
<tr>
<td>2004</td>
<td>−15.00</td>
<td>+10.00</td>
</tr>
<tr>
<td>2005</td>
<td>+0.60</td>
<td>−0.70</td>
</tr>
<tr>
<td>2006</td>
<td>+0.60</td>
<td>−0.70</td>
</tr>
<tr>
<td>2007</td>
<td>+0.60</td>
<td>−0.70</td>
</tr>
<tr>
<td>2008</td>
<td>+0.60</td>
<td>−0.70</td>
</tr>
<tr>
<td>2009</td>
<td>+15.60</td>
<td>−10.70</td>
</tr>
</tbody>
</table>
Typical Uses of a Currency Swap

- Conversion from a liability in one currency to a liability in another currency
- Conversion from an investment in one currency to an investment in another currency
## Comparative Advantage Arguments for Currency Swaps

(Table 7.8, page 167)

General Motors wants to borrow AUD
Qantas wants to borrow USD

<table>
<thead>
<tr>
<th></th>
<th>USD</th>
<th>AUD</th>
</tr>
</thead>
<tbody>
<tr>
<td>General Motors</td>
<td>5.0%</td>
<td>12.6%</td>
</tr>
<tr>
<td>Qantas</td>
<td>7.0%</td>
<td>13.0%</td>
</tr>
</tbody>
</table>
Valuation of Currency Swaps

Like interest rate swaps, currency swaps can be valued either as the difference between 2 bonds or as a portfolio of forward contracts.
Swaps & Forwards

- A swap can be regarded as a convenient way of packaging forward contracts.
- The “plain vanilla” interest rate swap in our example (slide 7.4) consisted of 6 FRAs.
- The “fixed for fixed” currency swap in our example (slide 7.22) consisted of a cash transaction & 5 forward contracts.
Swaps & Forwards (continued)

- The value of the swap is the sum of the values of the forward contracts underlying the swap
- Swaps are normally “at the money” initially
  - This means that it costs nothing to enter into a swap
  - It does not mean that each forward contract underlying a swap is “at the money” initially
Credit Risk

- A swap is worth zero to a company initially.
- At a future time its value is liable to be either positive or negative.
- The company has credit risk exposure only when its value is positive.
Other Types of Swaps

Floating-for-floating interest rate swaps, amortizing swaps, step up swaps, forward swaps, constant maturity swaps, compounding swaps, LIBOR-in-arrears swaps, accrual swaps, diff swaps, cross currency interest rate swaps, equity swaps, extendable swaps, puttable swaps, swaptions, commodity swaps, volatility swaps…….
Review of Option Types

- A call is an option to buy
- A put is an option to sell
- A European option can be exercised only at the end of its life
- An American option can be exercised at any time
Option Positions

- Long call
- Long put
- Short call
- Short put
Long Call on eBay
(Figure 8.1, Page 182)

Profit from buying one eBay European call option: option price = $5, strike price = $100, option life = 2 months
Short Call on eBay
(Figure 8.3, page 184)

Profit from writing one eBay European call option: option price = $5, strike price = $100
Long Put on IBM
(Figure 8.2, page 183)

Profit from buying an IBM European put option: option price = $7, strike price = $70
Short Put on IBM
(Figure 8.4, page 184)

Profit from writing an IBM European put option: option price = $7, strike price = $70
Payoffs from Options
What is the Option Position in Each Case?

\( K = \) Strike price, \( S_T = \) Price of asset at maturity

\[ \text{Payoff} \]

\[ \begin{array}{c}
\text{Payoff} \\
\uparrow \\
K \\
\downarrow \\
S_T
\end{array} \]

\[ \text{Payoff} \]

\[ \begin{array}{c}
\text{Payoff} \\
\uparrow \\
K \\
\downarrow \\
S_T
\end{array} \]
Assets Underlying Exchange-Traded Options

- Stocks
- Foreign Currency
- Stock Indices
- Futures
Specification of Exchange-Traded Options

- Expiration date
- Strike price
- European or American
- Call or Put (option class)
Terminology

Moneyness:
- At-the-money option
- In-the-money option
- Out-of-the-money option
Terminology (continued)

- Option class
- Option series
- Intrinsic value
- Time value
Dividends & Stock Splits
(Page 188-190)

- Suppose you own \( N \) options with a strike price of \( K \):
  - No adjustments are made to the option terms for cash dividends
  - When there is an \( n \)-for-\( m \) stock split,
    - the strike price is reduced to \( mK/n \)
    - the no. of options is increased to \( nN/m \)
  - Stock dividends are handled in a manner similar to stock splits
Consider a call option to buy 100 shares for $20/share.

How should terms be adjusted:

- for a 2-for-1 stock split?
- for a 5% stock dividend?
Market Makers

- Most exchanges use market makers to facilitate options trading
- A market maker quotes both bid and ask prices when requested
- The market maker does not know whether the individual requesting the quotes wants to buy or sell
Margins (Page 194-195)

- Margins are required when options are sold
- When a naked option is written the margin is the greater of:
  1. A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount (if any) by which the option is out of the money
  2. A total of 100% of the proceeds of the sale plus 10% of the underlying share price
- For other trading strategies there are special rules
Warrants

- Warrants are options that are issued by a corporation or a financial institution.
- The number of warrants outstanding is determined by the size of the original issue and changes only when they are exercised or when they expire.
Warrants (continued)

- The issuer settles up with the holder when a warrant is exercised
- When call warrants are issued by a corporation on its own stock, exercise will lead to new treasury stock being issued
Executive Stock Options

- Executive stock options are a form of remuneration issued by a company to its executives.
- They are usually at the money when issued.
- When options are exercised the company issues more stock and sells it to the option holder for the strike price.
Executive Stock Options continued

- They become vested after a period of time (usually 1 to 4 years)
- They cannot be sold
- They often last for as long as 10 or 15 years
- Accounting standards now require the expensing of executive stock options
Convertible Bonds

- Convertible bonds are regular bonds that can be exchanged for equity at certain times in the future according to a predetermined exchange ratio.
- Very often a convertible is callable.
- The call provision is a way in which the issuer can force conversion at a time earlier than the holder might otherwise choose.
Notation

- $c$: European call option price
- $p$: European put option price
- $S_0$: Stock price today
- $K$: Strike price
- $T$: Life of option
- $\sigma$: Volatility of stock price
- $C$: American Call option price
- $P$: American Put option price
- $S_T$: Stock price at option maturity
- $D$: Present value of dividends during option’s life
- $r$: Risk-free rate for maturity $T$ with cont comp
### Effect of Variables on Option Pricing

(Table 9.1, page 206)

<table>
<thead>
<tr>
<th>Variable</th>
<th>$c$</th>
<th>$p$</th>
<th>$C$</th>
<th>$P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$S_0$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$K$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
<tr>
<td>$T$</td>
<td>$?$</td>
<td>$?$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
<td>$+$</td>
</tr>
<tr>
<td>$r$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
</tr>
<tr>
<td>$D$</td>
<td>$-$</td>
<td>$+$</td>
<td>$-$</td>
<td>$+$</td>
</tr>
</tbody>
</table>
An American option is worth at least as much as the corresponding European option

\[ C \geq c \]

\[ P \geq p \]
Calls: An Arbitrage Opportunity?

Suppose that

\[ c = 3 \]
\[ S_0 = 20 \]
\[ T = 1 \]
\[ r = 10\% \]
\[ K = 18 \]
\[ D = 0 \]

Is there an arbitrage opportunity?
Lower Bound for European Call Option Prices; No Dividends (Equation 9.1, page 211)

\[ c \geq S_0 - Ke^{-rT} \]
Puts: An Arbitrage Opportunity?

- Suppose that
  \[ p = 1 \quad S_0 = 37 \]
  \[ T = 0.5 \quad r = 5\% \]
  \[ K = 40 \quad D = 0 \]

- Is there an arbitrage opportunity?
Lower Bound for European Put Prices; No Dividends

(Equation 9.2, page 212)

\[ p \geq Ke^{-rT} - S_0 \]
Put-Call Parity; No Dividends
(Equation 9.3, page 212)

- Consider the following 2 portfolios:
  - Portfolio A: European call on a stock + PV of the strike price in cash
  - Portfolio C: European put on the stock + the stock
- Both are worth \( \max(S_T, K) \) at the maturity of the options
- They must therefore be worth the same today. This means that

\[
c + Ke^{-rT} = p + S_0
\]
Arbitrage Opportunities

- Suppose that
  \[ c = 3 \quad S_0 = 31 \]
  \[ T = 0.25 \quad r = 10\% \]
  \[ K = 30 \quad D = 0 \]

- What are the arbitrage possibilities when
  \[ p = 2.25 \, ? \]
  \[ p = 1 \, ? \]
Early Exercise

- Usually there is some chance that an American option will be exercised early.
- An exception is an American call on a non-dividend paying stock.
- This should never be exercised early.
An Extreme Situation

- For an American call option:
  \( S_0 = 100; \ T = 0.25; \ K = 60; \ D = 0 \)
  Should you exercise immediately?
- What should you do if
  you want to hold the stock for the next 3 months?
  you do not feel that the stock is worth holding for the next 3 months?
9.13 Reasons For Not Exercising a Call Early (No Dividends)

- No income is sacrificed
- Payment of the strike price is delayed
- Holding the call provides insurance against stock price falling below strike price
Should Puts Be Exercised Early?

Are there any advantages to exercising an American put when

\[ S_0 = 60; \ T = 0.25; \ r = 10\% \]
\[ K = 100; \ D = 0 \]
The Impact of Dividends on Lower Bounds to Option Prices
(Equations 9.5 and 9.6, pages 218-219)

\[ c \geq S_0 - D - K e^{-rT} \]

\[ p \geq D + K e^{-rT} - S_0 \]
Extensions of Put-Call Parity

- American options; \( D = 0 \)
  \[ S_0 - K < C - P < S_0 - Ke^{-rT} \]
  (Equation 9.4, p. 215)

- European options; \( D > 0 \)
  \[ c + D + Ke^{-rT} = p + S_0 \]
  (Equation 9.7, p. 219)

- American options; \( D > 0 \)
  \[ S_0 - D - K < C - P < S_0 - Ke^{-rT} \]
  (Equation 9.8, p. 219)
Trading Strategies Involving Options

Chapter 10
Types of Strategies

- Take a position in the option and the underlying
- Take a position in 2 or more options of the same type (A spread)
- Combination: Take a position in a mixture of calls & puts (A combination)
Positions in an Option & the Underlying (Figure 10.1, page 224)

(a) (b) (c) (d)

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Bull Spread Using Calls
(Figure 10.2, page 225)
Bull Spread Using Puts

Figure 10.3, page 226
Bear Spread Using Puts

Figure 10.4, page 227
Bear Spread Using Calls

Figure 10.5, page 229

Profit

$K_1$ $K_2$ $S_T$
Box Spread

- A combination of a bull call spread and a bear put spread
- If all options are European a box spread is worth the present value of the difference between the strike prices
- If they are American this is not necessarily so. (See Business Snapshot 10.1)
Butterfly Spread Using Calls
Figure 10.6, page 231
Butterfly Spread Using Puts

Figure 10.7, page 232
Calendar Spread Using Calls
Figure 10.8, page 232

Profit

ST

K

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Calendar Spread Using Puts

Figure 10.9, page 233
A Straddle Combination

Figure 10.10, page 234

Profit

$K$

$S_T$
Strip & Strap

Figure 10.11, page 235

Profit

Strip

Profit

Strap

$K \quad S_T$

$K \quad S_T$
A Strangle Combination

Figure 10.12, page 236
Binomial Trees

Chapter 11
A Simple Binomial Model

- A stock price is currently $20
- In three months it will be either $22 or $18
A Call Option (Figure 11.1, page 242)

A 3-month call option on the stock has a strike price of 21.

- Stock price = $20
  - Option Price = $1
- Stock price = $22
  - Option Price = $1
- Stock price = $18
  - Option Price = $0

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Consider the Portfolio: long $\Delta$ shares
short 1 call option

- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$
Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:
  long 0.25 shares
  short 1 call option

- The value of the portfolio in 3 months is
  \[22 \times 0.25 - 1 = 4.50\]

- The value of the portfolio today is
  \[4.5e^{-0.12\times0.25} = 4.3670\]
Valuing the Option

- The portfolio that is long 0.25 shares short 1 option is worth 4.367.
- The value of the shares is 5.000 (\(= 0.25 \times 20\)).
- The value of the option is therefore 0.633 (\(= 5.000 - 4.367\)).
Generalization (Figure 11.2, page 243)

A derivative lasts for time $T$ and is dependent on a stock

\[ S_0, f, S_0u, f_u, S_0d, f_d \]
Generalization (continued)

- Consider the portfolio that is long $\Delta$ shares and short 1 derivative

$$S_0u\Delta - f_u \quad \quad S_0d\Delta - f_d$$

- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$
Generalization
(continued)

- Value of the portfolio at time $T$ is $S_0 u \Delta - f_u$
- Value of the portfolio today is $(S_0 u \Delta - f_u) e^{-rT}$
- Another expression for the portfolio value today is $S_0 \Delta - f$
- Hence $f = S_0 \Delta - (S_0 u \Delta - f_u) e^{-rT}$
Generalization (continued)

- Substituting for $\Delta$ we obtain

$$f = [ pf_u + (1 - p)f_d ] e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$
It is natural to interpret $p$ and $1-p$ as probabilities of up and down movements.

The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate.
Risk-neutral Valuation

- When the probability of an up and down movements are $p$ and $1-p$ the expected stock price at time $T$ is $S_0e^{rT}$
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation
Original Example Revisited

- Since $p$ is the probability that gives a return on the stock equal to the risk-free rate. We can find it from $20e^{0.12 \times 0.25} = 22p + 18(1 - p)$
  - which gives $p = 0.6523$
- Alternatively, we can use the formula
  \[
  p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12 \times 0.25} - 0.9}{1.1 - 0.9} = 0.6523
  \]
Valuing the Option Using Risk-Neutral Valuation

The value of the option is

$$e^{-0.12 \times 0.25} \times (0.6523 \times 1 + 0.3477 \times 0) = 0.633$$
Irrelevance of Stock’s Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant.
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant.
A Two-Step Example
Figure 11.3, page 246

Each time step is 3 months
K=21, r=12%
Valuing a Call Option
Figure 11.4, page 247

- Value at node B
  \[ e^{-0.12 \times 0.25} \times (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257 \]
- Value at node A
  \[ e^{-0.12 \times 0.25} \times (0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823 \]
A Put Option Example; $K=52$

Figure 11.7, page 250

$K = 52$, time step = 1yr

$r = 5\%$

![Diagram showing the put option example with values and labels.](image)
What Happens When an Option is American (Figure 11.8, page 251)
Delta

- Delta ($\Delta$) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock.
- The value of $\Delta$ varies from node to node.
Choosing $u$ and $d$

One way of matching the volatility is to set

$$u = e^{\sigma \sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma \sqrt{\Delta t}}$$

where $\sigma$ is the volatility and $\Delta t$ is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein.
The Probability of an Up Move

\[ p = \frac{a - d}{u - d} \]

\( a = e^{r \Delta t} \) for a nondividend paying stock

\( a = e^{(r-q) \Delta t} \) for a stock index where \( q \) is the dividend yield on the index

\( a = e^{(r-r_f) \Delta t} \) for a currency where \( r_f \) is the foreign risk-free rate

\( a = 1 \) for a futures contract
Wiener Processes and
Itô’s Lemma

Chapter 12
Types of Stochastic Processes

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable
Modeling Stock Prices

- We can use any of the four types of stochastic processes to model stock prices.
- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives.
Markov Processes (See pages 263-64)

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes
Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words, technical analysis does not work.

- A Markov process for stock prices is clearly consistent with weak-form market efficiency.
Example of a Discrete Time Continuous Variable Model

- A stock price is currently at $40
- At the end of 1 year it is considered that it will have a probability distribution of $\phi(40,10)$ where $\phi(\mu,\sigma)$ is a normal distribution with mean $\mu$ and standard deviation $\sigma$. 

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Questions

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- $\Delta t$ years?

Taking limits we have defined a continuous variable, continuous time process
Variance & Standard Deviations

- In Markov processes changes in successive periods of time are independent.
- This means that variances are additive.
- Standard deviations are not additive.
Variances & Standard Deviations
(continued)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.
A Wiener Process  

- We consider a variable $z$ whose value changes continuously
- The change in a small interval of time $\Delta t$ is $\Delta z$
- The variable follows a Wiener process if

1. $\Delta z = \varepsilon \sqrt{\Delta t}$ where $\varepsilon$ is $\phi(0,1)$

2. The values of $\Delta z$ for any 2 different (non-overlapping) periods of time are independent
Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is $T$
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$
What does an expression involving $dz$ and $dt$ mean?  
It should be interpreted as meaning that the corresponding expression involving $\Delta z$ and $\Delta t$ is true in the limit as $\Delta t$ tends to zero.  
In this respect, stochastic calculus is analogous to ordinary calculus.
A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1.

In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants.
The variable $x$ follows a generalized Wiener process with a drift rate of $a$ and a variance rate of $b^2$ if

$$dx = a \, dt + b \, dz$$
Generalized Wiener Processes (continued)

\[ \Delta x = a \Delta t + b \varepsilon \sqrt{\Delta t} \]

- Mean change in \( x \) in time \( T \) is \( aT \)
- Variance of change in \( x \) in time \( T \) is \( b^2 T \)
- Standard deviation of change in \( x \) in time \( T \) is \( b \sqrt{T} \)
The Example Revisited

- A stock price starts at 40 and has a probability distribution of $\phi(40,10)$ at the end of the year.
- If we assume the stochastic process is Markov with no drift, then the process is
  \[ dS = 10dz \]
- If the stock price were expected to grow by $8 on average during the year, so that the year-end distribution is $\phi(48,10)$, the process would be
  \[ dS = 8dt + 10dz \]
Itô Process (See pages 269)

- In an Itô process the drift rate and the variance rate are functions of time

\[ dx = a(x,t) \, dt + b(x,t) \, dz \]

- The discrete time equivalent

\[ \Delta x = a(x,t) \Delta t + b(x,t)\varepsilon \sqrt{\Delta t} \]

is only true in the limit as \( \Delta t \) tends to zero
Why a Generalized Wiener Process is not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time.
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price.
An Ito Process for Stock Prices
(See pages 269-71)

\[ dS = \mu S \, dt + \sigma S \, dz \]

where \( \mu \) is the expected return \( \sigma \) is the volatility.

The discrete time equivalent is

\[ \Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t} \]
Monte Carlo Simulation

- We can sample random paths for the stock price by sampling values for $\varepsilon$
- Suppose $\mu = 0.14$, $\sigma = 0.20$, and $\Delta t = 0.01$, then

$$\Delta S = 0.0014 S + 0.02 S \varepsilon$$
Monte Carlo Simulation – One Path (See Table 12.1, page 272)

<table>
<thead>
<tr>
<th>Period</th>
<th>Stock Price at Start of Period</th>
<th>Random Sample for $\varepsilon$</th>
<th>Change in Stock Price, $\Delta S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>20.000</td>
<td>0.52</td>
<td>0.236</td>
</tr>
<tr>
<td>1</td>
<td>20.236</td>
<td>1.44</td>
<td>0.611</td>
</tr>
<tr>
<td>2</td>
<td>20.847</td>
<td>-0.86</td>
<td>-0.329</td>
</tr>
<tr>
<td>3</td>
<td>20.518</td>
<td>1.46</td>
<td>0.628</td>
</tr>
<tr>
<td>4</td>
<td>21.146</td>
<td>-0.69</td>
<td>-0.262</td>
</tr>
</tbody>
</table>
Itô’s Lemma  (See pages 273-274)

- If we know the stochastic process followed by $x$, Itô’s lemma tells us the stochastic process followed by some function $G(x, t)$.
- Since a derivative security is a function of the price of the underlying and time, Itô’s lemma plays an important part in the analysis of derivative securities.
A Taylor’s series expansion of $G(x, t)$ gives

$$
\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2 \\
+ \frac{\partial^2 G}{\partial x \partial t} \Delta x \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial t^2} \Delta t^2 + \ldots
$$
Ignoring Terms of Higher Order Than $\Delta t$

In ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t$$

In stochastic calculus this becomes

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} \Delta x^2$$

because $\Delta x$ has a component which is of order $\sqrt{\Delta t}$
Substituting for $\Delta x$

Suppose
\[ dx = a(x, t)dt + b(x, t)dz \]
so that
\[ \Delta x = a \Delta t + b \epsilon \sqrt{\Delta t} \]

Then ignoring terms of higher order than $\Delta t$

\[ \Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \epsilon^2 \Delta t \]
The $\varepsilon^2 \Delta t$ Term

Since $\varepsilon \approx \phi(0,1)$, $E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \Delta t) = \Delta t$

The variance of $\Delta t$ is proportional to $\Delta t^2$ and can be ignored. Hence

$$\Delta G = \frac{\partial G}{\partial x} \Delta x + \frac{\partial G}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \Delta t$$
Taking Limits

Taking limits
\[ dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 dt \]

Substituting
\[ dx = a \, dt + b \, dz \]

We obtain
\[ dG = \left( \frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b \, dz \]

This is Ito's Lemma
Application of Ito’s Lemma to a Stock Price Process

The stock price process is

\[ dS = \mu S \, dt + \sigma S \, dz \]

For a function \( G \) of \( S \) and \( t \)

\[ dG = \left( \frac{\partial G}{\partial S} \mu S + \frac{\partial G}{\partial t} + \frac{1}{2} \frac{\partial^2 G}{\partial S^2} \sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S} \sigma S \, dz \]
Examples

1. The forward price of a stock for a contract maturing at time $T$

$$ G = S \ e^{r(T-t)} $$

$$ dG = (\mu - r)G \ dt + \sigma G \ dz $$

2. $G = \ln S$

$$ dG = \left( \mu - \frac{\sigma^2}{2} \right) dt + \sigma \ dz $$
The Black-Scholes-Merton Model

Chapter 13
The Stock Price Assumption

- Consider a stock whose price is $S$
- In a short period of time of length $\Delta t$, the return on the stock is normally distributed:

\[
\frac{\Delta S}{S} \approx \phi(\mu \Delta t, \sigma \sqrt{\Delta t})
\]

where $\mu$ is expected return and $\sigma$ is volatility
The Lognormal Property
(Equations 13.2 and 13.3, page 282)

- It follows from this assumption that

\[
\ln S_T - \ln S_0 \approx \phi \left[ \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]
\]

or

\[
\ln S_T \approx \phi \left[ \ln S_0 + \left( \mu - \frac{\sigma^2}{2} \right) T, \sigma \sqrt{T} \right]
\]

- Since the logarithm of \( S_T \) is normal, \( S_T \) is lognormally distributed
The Lognormal Distribution

\[ E(S_T) = S_0 \ e^{\mu T} \]

\[ \text{var}(S_T) = S_0^2 \ e^{2\mu T} (e^{\sigma^2 T} - 1) \]
Continuously Compounded Return, $x$
(Equations 13.6 and 13.7), page 283)

\[ S_T = S_0 e^{xT} \]

or

\[ x = \frac{1}{T} \ln \frac{S_T}{S_0} \]

or

\[ x \approx \phi \left( \mu - \frac{\sigma^2}{2}, \frac{\sigma}{\sqrt{T}} \right) \]
The Expected Return

- The expected value of the stock price is $S_0e^{\mu T}$
- The expected return on the stock is $\mu - \sigma^2/2$ not $\mu$

This is because

$$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

are not the same
μ and μ- σ²/2

Suppose we have daily data for a period of several months

- μ is the average of the returns in each day $[=E(\Delta S/S)]$
- μ- σ²/2 is the expected return over the whole period covered by the data measured with continuous compounding (or daily compounding, which is almost the same)
Mutual Fund Returns  (See Business Snapshot 13.1 on page 285)

- Suppose that returns in successive years are 15%, 20%, 30%, -20% and 25%
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4%
The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year.
- The standard deviation of the return in time $\Delta t$ is $\sigma \sqrt{\Delta t}$.
- If a stock price is $50 and its volatility is 25% per year what is the standard deviation of the price change in one day?
Estimating Volatility from Historical Data (page 286-88)

- Take observations $S_0, S_1, \ldots, S_n$ at intervals of $\tau$ years
- Calculate the continuously compounded return in each interval as:
  \[ u_i = \ln \left( \frac{S_i}{S_{i-1}} \right) \]
- Calculate the standard deviation, $s$, of the $u_i$'s
- The historical volatility estimate is:
  \[ \hat{\sigma} = \frac{s}{\sqrt{\tau}} \]
Nature of Volatility

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed.
- For this reason time is usually measured in “trading days” not calendar days when options are valued.
The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty.
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty.
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate.
- This leads to the Black-Scholes differential equation.
The Derivation of the Black-Scholes Differential Equation

\[ \Delta S = \mu S \Delta t + \sigma S \Delta z \]

\[ \Delta f = \left( \frac{\partial f}{\partial S} \mu S + \frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 \right) \Delta t + \frac{\partial f}{\partial S} \sigma S \Delta z \]

We set up a portfolio consisting of

-1: derivative

+ \frac{\partial f}{\partial S}: shares
The value of the portfolio \( \Pi \) is given by
\[
\Pi = -f + \frac{\partial f}{\partial S} S
\]
The change in its value in time \( \Delta t \) is given by
\[
\Delta \Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S
\]
The return on the portfolio must be the risk-free rate. Hence

$$\Delta \Pi = r \Pi \Delta t$$

We substitute for $\Delta f$ and $\Delta S$ in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$
The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is \( f = S - K \) when \( t = T \)
- The solution to the equation is \( f = S - K e^{-r(T-t)} \)
The Black-Scholes Formulas
(See pages 295-297)

\[
c = S_0 \, N(d_1) - K \, e^{-rT} \, N(d_2)
\]
\[
p = K \, e^{-rT} \, N(-d_2) - S_0 \, N(-d_1)
\]

where \(d_1 = \frac{\ln(S_0 / K) + (r + \sigma^2 / 2)T}{\sigma \sqrt{T}}\) and \(d_2 = \frac{\ln(S_0 / K) + (r - \sigma^2 / 2)T}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}\)
The \( N(x) \) Function

- \( N(x) \) is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than \( x \)
- See tables at the end of the book
Properties of Black-Scholes Formula

- As $S_0$ becomes very large $c$ tends to $S - Ke^{-rT}$ and $p$ tends to zero

- As $S_0$ becomes very small $c$ tends to zero and $p$ tends to $Ke^{-rT} - S$
Risk-Neutral Valuation

- The variable \( \mu \) does not appear in the Black-Scholes equation.
- The equation is independent of all variables affected by risk preference.
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world.
- This leads to the principle of risk-neutral valuation.
Applying Risk-Neutral Valuation
(See appendix at the end of Chapter 13)

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate
Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S e^{rT} - K$
- Present value of expected payoff is $e^{-rT}[S e^{rT} - K] = S - Ke^{-rT}$
Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price.
- There is a one-to-one correspondence between prices and implied volatilities.
- Traders and brokers often quote implied volatilities rather than dollar prices.
An Issue of Warrants & Executive Stock Options

- When a regular call option is exercised the stock that is delivered must be purchased in the open market.
- When a warrant or executive stock option is exercised new Treasury stock is issued by the company.
- If little or no benefits are foreseen by the market the stock price will reduce at the time the issue of is announced.
- There is no further dilution (See Business Snapshot 13.3.)
The Impact of Dilution

- After the options have been issued it is not necessary to take account of dilution when they are valued.

- Before they are issued we can calculate the cost of each option as $\frac{N}{N+M}$ times the price of a regular option with the same terms where $N$ is the number of existing shares and $M$ is the number of new shares that will be created if exercise takes place.
Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes.

- Only dividends with ex-dividend dates during the life of the option should be included.

- The “dividend” should be the expected reduction in the stock price expected.
American Calls

- An American call on a non-dividend-paying stock should never be exercised early.
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date.
- Suppose dividend dates are at times $t_1$, $t_2$, $\ldots t_n$. Early exercise is sometimes optimal at time $t_i$ if the dividend at that time is greater than

$$K[1 - e^{-r(t_{i+1} - t_i)}]$$
Black’s Approximation for Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:
1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date
Options on Stock Indices, Currencies, and Futures

Chapter 14
European Options on Stocks Providing a Dividend Yield

We get the same probability distribution for the stock price at time $T$ in each of the following cases:

1. The stock starts at price $S_0$ and provides a dividend yield $= q$

2. The stock starts at price $S_0 e^{-QT}$ and provides no income
European Options on Stocks
Providing Dividend Yield
continued

We can value European options by reducing the stock price to $S_0e^{-qT}$ and then behaving as though there is no dividend.
Extension of Chapter 9 Results
(Equations 14.1 to 14.3)

Lower Bound for calls:

\[ c \geq S_0 e^{-qT} - Ke^{-rT} \]

Lower Bound for puts

\[ p \geq Ke^{-rT} - S_0 e^{-qT} \]

Put Call Parity

\[ c + Ke^{-rT} = p + S_0 e^{-qT} \]
Extension of Chapter 13 Results
(Equations 14.4 and 14.5)

\[ c = S_0 e^{-qT} N(d_1) - Ke^{-rT} N(d_2) \]
\[ p = Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \]

where
\[ d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma \sqrt{T}} \]
\[ d_2 = \frac{\ln(S_0 / K) + (r - q - \sigma^2 / 2)T}{\sigma \sqrt{T}} \]
The Binomial Model

\[ f = e^{-rT} \left[ pf_u + (1-p)f_d \right] \]

\[ S_0 \]

\[ p \]

\[ S_0u \]

\[ f_u \]

\[ (1-p) \]

\[ S_0d \]

\[ f_d \]
The Binomial Model
continued

- In a risk-neutral world the stock price grows at $r-q$ rather than at $r$ when there is a dividend yield at rate $q$

- The probability, $p$, of an up movement must therefore satisfy

$$ pS_0u + (1-p)S_0d = S_0e^{(r-q)T} $$

so that

$$ p = \frac{e^{(r-q)T} - d}{u - d} $$
Index Options  (page 316-321)

- The most popular underlying indices in the U.S. are
  - The Dow Jones Index times 0.01 (DJX)
  - The Nasdaq 100 Index (NDX)
  - The Russell 2000 Index (RUT)
  - The S&P 100 Index (OEX)
  - The S&P 500 Index (SPX)
- Contracts are on 100 times index; they are settled in cash; OEX is American and the rest are European.
LEAPS

- Leaps are options on stock indices that last up to 3 years
- They have December expiration dates
- They are on 10 times the index
- Leaps also trade on some individual stocks
Index Option Example

- Consider a call option on an index with a strike price of 560
- Suppose 1 contract is exercised when the index level is 580
- What is the payoff?
Using Index Options for Portfolio Insurance

- Suppose the value of the index is $S_0$ and the strike price is $K$.
- If a portfolio has a $\beta$ of 1.0, the portfolio insurance is obtained by buying 1 put option contract on the index for each $100S_0$ dollars held.
- If the $\beta$ is not 1.0, the portfolio manager buys $\beta$ put options for each $100S_0$ dollars held.
- In both cases, $K$ is chosen to give the appropriate insurance level.
Example 1

- Portfolio has a beta of 1.0
- It is currently worth $500,000
- The index currently stands at 1000
- What trade is necessary to provide insurance against the portfolio value falling below $450,000?
Example 2

- Portfolio has a beta of 2.0
- It is currently worth $500,000 and index stands at 1000
- The risk-free rate is 12% per annum
- The dividend yield on both the portfolio and the index is 4%
- How many put option contracts should be purchased for portfolio insurance?
Calculating Relation Between Index Level and Portfolio Value in 3 months

- If index rises to 1040, it provides a 40/1000 or 4% return in 3 months
- Total return (incl dividends)=5%
- Excess return over risk-free rate=2%
- Excess return for portfolio=4%
- Increase in Portfolio Value=4+3-1=6%
- Portfolio value=$530,000
### Determining the Strike Price (Table 14.2, page 320)

<table>
<thead>
<tr>
<th>Value of Index in 3 months</th>
<th>Expected Portfolio Value in 3 months ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1,080</td>
<td>570,000</td>
</tr>
<tr>
<td>1,040</td>
<td>530,000</td>
</tr>
<tr>
<td>1,000</td>
<td>490,000</td>
</tr>
<tr>
<td>960</td>
<td>450,000</td>
</tr>
<tr>
<td>920</td>
<td>410,000</td>
</tr>
</tbody>
</table>

An option with a strike price of 960 will provide protection against a 10% decline in the portfolio value.
Valuing European Index Options

We can use the formula for an option on a stock paying a dividend yield
Set $S_0 = \text{current index level}$
Set $q = \text{average dividend yield expected during the life of the option}$
Currency Options

- Currency options trade on the Philadelphia Exchange (PHLX)
- There also exists an active over-the-counter (OTC) market
- Currency options are used by corporations to buy insurance when they have an FX exposure
The Foreign Interest Rate

- We denote the foreign interest rate by $r_f$
- When a U.S. company buys one unit of the foreign currency it has an investment of $S_0$ dollars
- The return from investing at the foreign rate is $r_f S_0$ dollars
- This shows that the foreign currency provides a “dividend yield” at rate $r_f$
Valuing European Currency Options

- A foreign currency is an asset that provides a “dividend yield” equal to $r_f$
- We can use the formula for an option on a stock paying a dividend yield:

\[
\text{Set } S_0 = \text{current exchange rate} \\
\text{Set } q = r_f
\]
Formulas for European Currency Options
(Equations 14.7 and 14.8, page 322)

\[
c = S_0 e^{-r_f T} N(d_1) - Ke^{-rT} N(d_2)
\]

\[
p = Ke^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1)
\]

where

\[
d_1 = \frac{\ln(S_0 / K) + (r - r_f + \sigma^2 / 2)T}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln(S_0 / K) + (r - r_f - \sigma^2 / 2)T}{\sigma \sqrt{T}}
\]
Alternative Formulas
(Equations 14.9 and 14.10, page 322)

Using

\[ F_0 = S_0 e^{(r - r_f)T} \]

\[ c = e^{-rT} \left[ F_0 N(d_1) - K N(d_2) \right] \]

\[ p = e^{-rT} \left[ K N(-d_2) - F_0 N(-d_1) \right] \]

\[ d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma \sqrt{T}} \]

\[ d_2 = d_1 - \sigma \sqrt{T} \]
Mechanics of Call Futures Options

When a call futures option is exercised the holder acquires
1. A long position in the futures
2. A cash amount equal to the excess of the futures price over the strike price
Mechanics of Put Futures Option

When a put futures option is exercised the holder acquires

1. A short position in the futures
2. A cash amount equal to the excess of the strike price over the futures price
The Payoffs

If the futures position is closed out immediately:

Payoff from call = $F_0 - K$

Payoff from put = $K - F_0$

where $F_0$ is futures price at time of exercise
Put-Call Parity for Futures Options  (Equation 14.11, page 329)

Consider the following two portfolios:
1. European call plus $Ke^{-rT}$ of cash
2. European put plus long futures plus cash equal to $F_0 e^{-rT}$

They must be worth the same at time $T$ so that

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$
Binomial Tree Example

A 1-month call option on futures has a strike price of 29.

Futures price = $30
Option Price = ?

Futures Price = $33
Option Price = $4

Futures Price = $28
Option Price = $0
Consider the Portfolio: long $\Delta$ futures
short 1 call option

- Portfolio is riskless when $3\Delta - 4 = -2\Delta$ or $\Delta = 0.8$
Valuing the Portfolio
( Risk-Free Rate is 6% )

- The riskless portfolio is:
  long 0.8 futures
  short 1 call option

- The value of the portfolio in 1 month is -1.6

- The value of the portfolio today is
  \(-1.6e^{-\frac{0.06}{12}} = -1.592\)
Valuing the Option

- The portfolio that is long 0.8 futures short 1 option is worth -1.592.
- The value of the futures is zero.
- The value of the option must therefore be 1.592.
A derivative lasts for time $T$ and is dependent on a futures price.
Generalization
(continued)

- Consider the portfolio that is long $\Delta$ futures and short 1 derivative

\[
\begin{align*}
F_0u \Delta - F_0 \Delta - f_u \\
F_0d \Delta - F_0 \Delta - f_d
\end{align*}
\]

- The portfolio is riskless when

\[
\Delta = \frac{f_u - f_d}{F_0u - F_0d}
\]
Generalization
(continued)

- Value of the portfolio at time $T$ is $F_0 u \Delta - F_0 \Delta - f_u$
- Value of portfolio today is $-f$
- Hence
  \[ f = - [F_0 u \Delta - F_0 \Delta - f_u] e^{-rT} \]
Generalization
(continued)

- Substituting for $\Delta$ we obtain

$$f = \left[ p \, f_u + (1 - p) \, f_d \right] e^{-rT}$$

where

$$p = \frac{1 - d}{u - d}$$
Valuing European Futures Options

- We can use the formula for an option on a stock paying a dividend yield
  
  \[ S_0 = \text{current futures price} \ (F_0) \]
  
  \[ q = \text{domestic risk-free rate} \ (r) \]

- Setting \( q = r \) ensures that the expected growth of \( F \) in a risk-neutral world is zero
Growth Rates For Futures Prices

- A futures contract requires no initial investment
- In a risk-neutral world the expected return should be zero
- The expected growth rate of the futures price is therefore zero
- The futures price can therefore be treated like a stock paying a dividend yield of $r$
Black’s Formula
(Equations 14.16 and 14.17, page 333)

- The formulas for European options on futures are known as Black’s formulas

\[
c = e^{-rT} \left[ F_0 \ N(d_1) - K \ N(d_2) \right]
\[
p = e^{-rT} \left[ K \ N(-d_2) - F_0 \ N(-d_1) \right]
\]

where

\[
d_1 = \frac{\ln\left(\frac{F_0}{K}\right) + \sigma^2 T / 2}{\sigma \sqrt{T}}
\]

\[
d_2 = \frac{\ln\left(\frac{F_0}{K}\right) - \sigma^2 T / 2}{\sigma \sqrt{T}} = d_1 - \sigma \sqrt{T}
\]
Futures Option Prices vs Spot Option Prices

- If futures prices are higher than spot prices (normal market), an American call on futures is worth more than a similar American call on spot. An American put on futures is worth less than a similar American put on spot.
- When futures prices are lower than spot prices (inverted market) the reverse is true.
Summary of Key Results

- We can treat stock indices, currencies, and futures like a stock paying a dividend yield of $q$
  - For stock indices, $q = \text{average dividend yield on the index over the option life}$
  - For currencies, $q = r_f$
  - For futures, $q = r$
The Greek Letters

Chapter 15
Example

- A bank has sold for $300,000 a European call option on 100,000 shares of a nondividend paying stock
- \( S_0 = 49, \quad K = 50, \quad r = 5\% , \quad \sigma = 20\% , \quad T = 20 \) weeks, \( \mu = 13\% \)
- The Black-Scholes value of the option is $240,000
- How does the bank hedge its risk to lock in a $60,000 profit?
Naked & Covered Positions

Naked position
   Take no action

Covered position
   Buy 100,000 shares today

Both strategies leave the bank exposed to significant risk
Stop-Loss Strategy

This involves:

- Buying 100,000 shares as soon as price reaches $50
- Selling 100,000 shares as soon as price falls below $50

This deceptively simple hedging strategy does not work well
Delta (See Figure 15.2, page 345)

- Delta ($\Delta$) is the rate of change of the option price with respect to the underlying stock price.

![Diagram showing the relationship between option price and stock price with a slope labeled as \( \Delta \).]
Delta Hedging

- This involves maintaining a delta neutral portfolio.
- The delta of a European call on a stock paying dividends at rate $q$ is $N(d_1)e^{-qT}$.
- The delta of a European put is $e^{-qT}[N(d_1) - 1]$. 
Delta Hedging continued

- The hedge position must be frequently rebalanced.
- Delta hedging a written option involves a “buy high, sell low” trading rule.
- See Tables 15.2 (page 350) and 15.3 (page 351) for examples of delta hedging.
Using Futures for Delta Hedging

- The delta of a futures contract is $e^{(r-q)T}$ times the delta of a spot contract.
- The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract.
Theta

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time.

- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of the option declines.
Gamma

- Gamma ($\Gamma$) is the rate of change of delta ($\Delta$) with respect to the price of the underlying asset.
- Gamma is greatest for options that are close to the money (see Figure 15.9, page 358).
Gamma Addresses Delta Hedging Errors Caused By Curvature
(Figure 15.7, page 355)
Interpretation of Gamma

- For a delta neutral portfolio,
  \[ \Delta \Pi \approx \Theta \Delta t + \frac{1}{2} \Gamma \Delta S^2 \]
For a portfolio of derivatives on a stock paying a continuous dividend yield at rate $q$

$$\Theta + (r - q)S\Delta + \frac{1}{2} \sigma^2 S^2 \Gamma = r \Pi$$
Vega

- Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility.
- Vega tends to be greatest for options that are close to the money (See Figure 15.11, page 361).
Managing Delta, Gamma, & Vega

- $\Delta$ can be changed by taking a position in the underlying.
- To adjust $\Gamma$ & $\nu$ it is necessary to take a position in an option or other derivative.
Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate

- For currency options there are 2 rhos
Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day.
- Whenever the opportunity arises, they improve gamma and vega.
- As portfolio becomes larger, hedging becomes less expensive.
Scenario Analysis

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities.
Hedging vs Creation of an Option Synthetically

- When we are hedging we take positions that offset $\Delta$, $\Gamma$, $\nu$, etc.
- When we create an option synthetically we take positions that match $\Delta$, $\Gamma$, & $\nu$
Portfolio Insurance

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically.
- This involves initially selling enough of the portfolio (or of index futures) to match the $\Delta$ of the put option.
Portfolio Insurance continued

- As the value of the portfolio increases, the $\Delta$ of the put becomes less negative and some of the original portfolio is repurchased.
- As the value of the portfolio decreases, the $\Delta$ of the put becomes more negative and more of the portfolio must be sold.
Portfolio Insurance continued

The strategy did not work well on October 19, 1987...
Volatility Smiles

Chapter 16
Put-Call Parity Arguments

- Put-call parity \( p + S_0 e^{-qT} = c + K e^{-rT} \) holds regardless of the assumptions made about the stock price distribution.

- It follows that

\[
\rho_{\text{mkt}} - \rho_{\text{bs}} = c_{\text{mkt}} - c_{\text{bs}}
\]
Implied Volatilities

- When $p_{bs}=p_{mkt}$, it must be true that $c_{bs}=c_{mkt}$
- It follows that the implied volatility calculated from a European call option should be the same as that calculated from a European put option when both have the same strike price and maturity
- The same is approximately true of American options
Volatility Smile

- A volatility smile shows the variation of the implied volatility with the strike price.
- The volatility smile should be the same whether calculated from call options or put options.
The Volatility Smile for Foreign Currency Options
(Figure 16.1, page 377)
Implied Distribution for Foreign Currency Options (Figure 16.2, page 377)

- Both tails are heavier than the lognormal distribution
- It is also “more peaked” than the lognormal distribution
The Volatility Smile for Equity Options (Figure 16.3, page 380)
The left tail is heavier and the right tail is less heavy than the lognormal distribution.
Other Volatility Smiles?

What is the volatility smile if

- True distribution has a less heavy left tail and heavier right tail
- True distribution has both a less heavy left tail and a less heavy right tail
Possible Causes of Volatility Smile

- Asset price exhibiting jumps rather than continuous change
- Volatility for asset price being stochastic

(One reason for a stochastic volatility in the case of equities is the relationship between volatility and leverage)
Volatility Term Structure

- In addition to calculating a volatility smile, traders also calculate a volatility term structure.
- This shows the variation of implied volatility with the time to maturity of the option.
Volatility Term Structure

The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low.
### Example of a Volatility Surface

(Table 16.2, page 382)

<table>
<thead>
<tr>
<th>Strike Price</th>
<th>0.90</th>
<th>0.95</th>
<th>1.00</th>
<th>1.05</th>
<th>1.10</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 month</td>
<td>14.2</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.5</td>
</tr>
<tr>
<td>3 month</td>
<td>14.0</td>
<td>13.0</td>
<td>12.0</td>
<td>13.1</td>
<td>14.2</td>
</tr>
<tr>
<td>6 month</td>
<td>14.1</td>
<td>13.3</td>
<td>12.5</td>
<td>13.4</td>
<td>14.3</td>
</tr>
<tr>
<td>1 year</td>
<td>14.7</td>
<td>14.0</td>
<td>13.5</td>
<td>14.0</td>
<td>14.8</td>
</tr>
<tr>
<td>2 year</td>
<td>15.0</td>
<td>14.4</td>
<td>14.0</td>
<td>14.5</td>
<td>15.1</td>
</tr>
<tr>
<td>5 year</td>
<td>14.8</td>
<td>14.6</td>
<td>14.4</td>
<td>14.7</td>
<td>15.0</td>
</tr>
</tbody>
</table>
Basic Numerical Procedures

Chapter 17
Tree Approaches to Derivatives Valuation

- Trees
- Monte Carlo simulation
- Finite difference methods
Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset.
- In each small interval of time the stock price is assumed to move up by a proportional amount $u$ or to move down by a proportional amount $d$. 
Movements in Time $\Delta t$
(Figure 17.1, page 392)
1. Tree Parameters for asset paying a dividend yield of $q$

Parameters $p$, $u$, and $d$ are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world.

Mean: 
\[ e^{(r-q)\Delta t} = pu + (1-p)d \]

Variance: 
\[ \sigma^2 \Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t} \]

A further condition often imposed is $u = 1/d$.
2. Tree Parameters for asset paying a dividend yield of $q$

(Equations 17.4 to 17.7)

When $\Delta t$ is small a solution to the equations is

\[
\begin{align*}
    u &= e^{\sigma \sqrt{\Delta t}} \\
    d &= e^{-\sigma \sqrt{\Delta t}} \\
    p &= \frac{a - d}{u - d} \\
    a &= e^{(r-q) \Delta t}
\end{align*}
\]
The Complete Tree
(Figure 17.2, page 394)
Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate
Example: Put Option
(Example 17.1, page 394)

\[ S_0 = 50; \quad K = 50; \quad r = 10\%; \quad \sigma = 40\%; \]
\[ T = 5 \text{ months} = 0.4167; \]
\[ \Delta t = 1 \text{ month} = 0.0833 \]

The parameters imply
\[ u = 1.1224; \quad d = 0.8909; \]
\[ a = 1.0084; \quad p = 0.5073 \]
Example (continued)
Figure 17.3, page 395
Calculation of Delta

Delta is calculated from the nodes at time $\Delta t$

$$\text{Delta} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$
Calculation of Gamma

Gamma is calculated from the nodes at time $2\Delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24; \quad \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$
Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\Delta t$

$$
\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}
$$

or -0.012 per calendar day
Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is

\[ 4.62 - 4.49 = 0.13 \]

per 1% change in volatility
Trees for Options on Indices, Currencies and Futures Contracts

As with Black-Scholes:

- For options on stock indices, $q$ equals the dividend yield on the index.
- For options on a foreign currency, $q$ equals the foreign risk-free rate.
- For options on futures contracts $q = r$. 

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Binomial Tree for Dividend Paying Stock

- Procedure:
  - Draw the tree for the stock price less the present value of the dividends
  - Create a new tree by adding the present value of the dividends at each node
  - This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes model is used
Extensions of Tree Approach

- Time dependent interest rates
- The control variate technique
Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

\[ u = e^{(r-q-\sigma^2/2)\Delta t + \sigma \sqrt{\Delta t}} \]

\[ d = e^{(r-q-\sigma^2/2)\Delta t - \sigma \sqrt{\Delta t}} \]
Trinomial Tree (Page 409)

\[ u = e^{\sigma\sqrt{3}\Delta t} \quad d = 1/u \]

\[ p_u = \sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6} \]

\[ p_m = \frac{2}{3} \]

\[ p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}} \left( r - \frac{\sigma^2}{2} \right) + \frac{1}{6} \]
Time Dependent Parameters in a Binomial Tree (page 409)

- Making \( r \) or \( q \) a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time.

- We can make \( \sigma \) a function of time by making the lengths of the time steps inversely proportional to the variance rate.
Monte Carlo Simulation and $\pi$

- How could you calculate $\pi$ by randomly sampling points in the square?
Monte Carlo Simulation and Options

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option
Sampling Stock Price Movements
(Equations 17.13 and 17.14, page 411)

- In a risk neutral world the process for a stock price is

\[ dS = \mu S \, dt + \sigma S \, dz \]

- We can simulate a path by choosing time steps of length \( \Delta t \) and using the discrete version of this

\[ \Delta S = \mu S \, \Delta t + \sigma S \, \varepsilon \sqrt{\Delta t} \]

where \( \varepsilon \) is a random sample from \( \phi(0,1) \)
A More Accurate Approach
(Equation 17.15, page 412)

Use

$$d \ln S = \left( \hat{\mu} - \sigma^2 / 2 \right) dt + \sigma \, dz$$

The discrete version of this is

$$\ln S(t + \Delta t) - \ln S(t) = \left( \hat{\mu} - \sigma^2 / 2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

or

$$S(t + \Delta t) = S(t) \, e^{\left( \hat{\mu} - \sigma^2 / 2 \right) \Delta t + \sigma \varepsilon \sqrt{\Delta t}}$$
Extensions

When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the values for the derivative.
One simple way to obtain a sample from $\phi(0,1)$ is to generate 12 random numbers between 0.0 & 1.0, take the sum, and subtract 6.0. In Excel, `=NORMSINV(RAND())` gives a random sample from $\phi(0,1)$.
To Obtain 2 Correlated Normal Samples

- Obtain independent normal samples $x_1$ and $x_2$ and set
  
  $\varepsilon_1 = x_1$
  
  $\varepsilon_2 = \rho x_1 + x_2 \sqrt{1 - \rho^2}$

- A procedure known as Cholesky’s decomposition when samples are required from more than two normal variables
Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.
Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs.

It cannot easily deal with American-style options.
Determining Greek Letters

For $\Delta$:

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate $\Delta$ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters
Variance Reduction Techniques

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Stratified sampling
- Moment matching
- Using quasi-random sequences
Sampling Through the Tree

Instead of sampling from the stochastic process we can sample paths randomly through a binomial or trinomial tree to value a derivative
Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation.
- We form a grid by considering equally spaced time values and stock price values.
- Define $f_{i,j}$ as the value of $f$ at time $i\Delta t$ when the stock price is $j\Delta S$. 
Finite Difference Methods
(continued)

\[
\frac{\partial f}{\partial t} + rS \frac{\partial f}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf
\]

we set
\[
\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}
\]

\[
\frac{\partial^2 f}{\partial S^2} = \left( \frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \right) / \Delta S \quad \text{or}
\]

\[
\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}
\]
Implicit Finite Difference Method  
(Equation 17.25, page 420)

If we also set \[ \frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t} \]
we obtain the implicit finite difference method. This involves solving simultaneous equations of the form:

\[ a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j} \]
Explicit Finite Difference Method
(page 422-428)

If \( \frac{\partial f}{\partial S} \) and \( \frac{\partial^2 f}{\partial S^2} \) are assumed to be the same at the \((i+1,j)\) point as they are at the \((i,j)\) point we obtain the explicit finite difference method. This involves solving equations of the form:

\[
f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}
\]
Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach
Implicit vs Explicit
Finite Difference Methods
(Figure 17.16, page 425)

Implicit Method

Explicit Method

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Other Points on Finite Difference Methods

- It is better to have \( \ln S \) rather than \( S \) as the underlying variable
- Improvements over the basic implicit and explicit methods:
  - Hopscotch method
  - Crank-Nicolson method
Value at Risk

Chapter 18
The Question Being Asked in VaR

“What loss level is such that we are $X\%$ confident it will not be exceeded in $N$ business days?”
VaR and Regulatory Capital
(Business Snapshot 18.1, page 436)

- Regulators base the capital they require banks to keep on VaR
- The market-risk capital is $k$ times the 10-day 99% VaR where $k$ is at least 3.0
VaR vs. C-VaR
(See Figures 18.1 and 18.2)

- VaR is the loss level that will not be exceeded with a specified probability.
- C-VaR (or expected shortfall) is the expected loss given that the loss is greater than the VaR level.
- Although C-VaR is theoretically more appealing, it is not widely used.
Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”
Time Horizon

- Instead of calculating the 10-day, 99% VaR directly analysts usually calculate a 1-day 99% VaR and assume

\[ 10\text{-day VaR} = \sqrt{10} \times 1\text{-day VaR} \]

- This is exactly true when portfolio changes on successive days come from independent identically distributed normal distributions
Historical Simulation
(See Tables 18.1 and 18.2, page 438-439))

- Create a database of the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day.
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day.
- and so on.
Historical Simulation continued

- Suppose we use $m$ days of historical data
- Let $v_i$ be the value of a variable on day $i$
- There are $m-1$ simulation trials
- The $i$th trial assumes that the value of the market variable tomorrow (i.e., on day $m+1$) is

$$v_m \frac{v_i}{v_{i-1}}$$
The Model-Building Approach

1. The main alternative to historical simulation is to make assumptions about the probability distributions of return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically.

2. This is known as the model building approach or the variance-covariance approach.
Daily Volatilities

- In option pricing we measure volatility “per year”
- In VaR calculations we measure volatility “per day”

\[ \sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}} \]
Daily Volatility continued

- Strictly speaking we should define $\sigma_{\text{day}}$ as the standard deviation of the continuously compounded return in one day.
- In practice we assume that it is the standard deviation of the percentage change in one day.
Microsoft Example (page 440)

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use $N=10$ and $X=99$
Microsoft Example continued

- The standard deviation of the change in the portfolio in 1 day is $200,000.
- The standard deviation of the change in 10 days is

\[ 200,000 \sqrt{10} = 632,456 \]
Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- We assume that the change in the value of the portfolio is normally distributed
- Since $N(-2.33)=0.01$, the VaR is

$$2.33 \times 632,456 = $1,473,621$$
AT&T Example (page 441)

- Consider a position of $5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The S.D per 10 days is
  \[
  50,000 \sqrt{10} = $158,144
  \]
- The VaR is
  \[
  158,114 \times 2.33 = $368,405
  \]
Portfolio

- Now consider a portfolio consisting of both Microsoft and AT&T.
- Suppose that the correlation between the returns is 0.3.
S.D. of Portfolio

- A standard result in statistics states that

\[ \sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho \sigma_X \sigma_Y} \]

- In this case \( \sigma_X = 200,000 \) and \( \sigma_Y = 50,000 \) and \( \rho = 0.3 \). The standard deviation of the change in the portfolio value in one day is therefore 220,227
**VaR for Portfolio**

- The 10-day 99% VaR for the portfolio is
  \[ 220,227 \times \sqrt{10} \times 2.33 = $1,622,657 \]
- The benefits of diversification are
  \[(1,473,621+368,405) - 1,622,657 = $219,369 \]
- What is the incremental effect of the AT&T holding on VaR?
The Linear Model

We assume

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed
The General Linear Model continued (equations 18.1 and 18.2)

\[
\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i
\]

\[
\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}
\]

\[
\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}
\]

where \(\sigma_i\) is the volatility of variable \(i\)
and \(\sigma_p\) is the portfolio's standard deviation.
Handling Interest Rates: Cash Flow Mapping

- We choose as market variables bond prices with standard maturities (1mth, 3mth, 6mth, 1yr, 2yr, 5yr, 7yr, 10yr, 30yr)
- Suppose that the 5yr rate is 6% and the 7yr rate is 7% and we will receive a cash flow of $10,000 in 6.5 years.
- The volatilities per day of the 5yr and 7yr bonds are 0.50% and 0.58% respectively
Example continued

- We interpolate between the 5yr rate of 6% and the 7yr rate of 7% to get a 6.5yr rate of 6.75%
- The PV of the $10,000 cash flow is

\[
\frac{10,000}{1.0675^{6.5}} = 6,540
\]
Example continued

- We interpolate between the 0.5% volatility for the 5yr bond price and the 0.58% volatility for the 7yr bond price to get 0.56% as the volatility for the 6.5yr bond.
- We allocate $\alpha$ of the PV to the 5yr bond and $(1 - \alpha)$ of the PV to the 7yr bond.
Example continued

- Suppose that the correlation between movement in the 5yr and 7yr bond prices is 0.6
- To match variances
  \[ 0.56^2 = 0.5^2 \alpha^2 + 0.58^2 (1-\alpha)^2 + 2 \times 0.6 \times 0.5 \times 0.58 \times \alpha (1-\alpha) \]
- This gives \( \alpha = 0.074 \)
Example continued

The value of 6,540 received in 6.5 years
\[ 6,540 \times 0.074 = 484 \]

in 5 years and by
\[ 6,540 \times 0.926 = 6,056 \]
in 7 years.

This cash flow mapping preserves value and variance.
When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap
Consider a portfolio of options dependent on a single stock price, $S$. Define

$$\delta = \frac{\Delta P}{\Delta S}$$

and

$$\Delta x = \frac{\Delta S}{S}$$
Linear Model and Options continued (equations 18.3 and 18.4)

- As an approximation
  \[ \Delta P = \delta \Delta S = S \delta \Delta x \]

- Similarly when there are many underlying market variables
  \[ \Delta P = \sum_i S_i \delta_i \Delta x_i \]

where \( \delta_i \) is the delta of the portfolio with respect to the \( i \)th asset.
Example

- Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are 120 and 30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively.

- As an approximation

\[ \Delta P = 120 \times 1,000 \Delta x_1 + 30 \times 20,000 \Delta x_2 \]

where \( \Delta x_1 \) and \( \Delta x_2 \) are the percentage changes in the two stock prices.
Skewness
(See Figures 18.3, 18.4, and 18.5)

The linear model fails to capture skewness in the probability distribution of the portfolio value.
Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

\[ \Delta P = \delta \Delta S + \frac{1}{2} \gamma (\Delta S)^2 \]

this becomes

\[ \Delta P = S \delta \Delta x + \frac{1}{2} S^2 \gamma (\Delta x)^2 \]
Quadratic Model continued

With many market variables we get an expression of the form

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where

$$\delta_i = \frac{\partial P}{\partial S_i} \quad \gamma_{ij} = \frac{\partial^2 P}{\partial S_i S_j}$$

This is not as easy to work with as the linear model.
Monte Carlo Simulation (page 448-449)

To calculate VaR using M.C. simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the $\Delta x_i$
- Use the $\Delta x_i$ to determine market variables at end of one day
- Revalue the portfolio at the end of day
Monte Carlo Simulation

- Calculate $\Delta P$
- Repeat many times to build up a probability distribution for $\Delta P$
- VaR is the appropriate fractile of the distribution times square root of $N$
- For example, with 1,000 trial the 1 percentile is the 10th worst case.
Speeding Up Monte Carlo

Use the quadratic approximation to calculate $\Delta P$
Comparison of Approaches

- Model building approach assumes normal distributions for market variables. It tends to give poor results for low delta portfolios.
- Historical simulation lets historical data determine distributions, but is computationally slower.
Stress Testing

- This involves testing how well a portfolio performs under some of the most extreme market moves seen in the last 10 to 20 years.
Back-Testing

- Tests how well VaR estimates would have performed in the past
- We could ask the question: How often was the actual 10-day loss greater than the 99%/10 day VaR?
Principal Components Analysis for Interest Rates (Tables 18.3 and 18.4 on page 451)

- The first factor is a roughly parallel shift (83.1% of variation explained)
- The second factor is a twist (10% of variation explained)
- The third factor is a bowing (2.8% of variation explained)
Using PCA to calculate VaR (page 453)

Example: Sensitivity of portfolio to rates ($m)

<table>
<thead>
<tr>
<th>1 yr</th>
<th>2 yr</th>
<th>3 yr</th>
<th>4 yr</th>
<th>5 yr</th>
</tr>
</thead>
<tbody>
<tr>
<td>+10</td>
<td>+4</td>
<td>-8</td>
<td>-7</td>
<td>+2</td>
</tr>
</tbody>
</table>

Sensitivity to first factor is from Table 18.3:

\[ 10 \times 0.32 + 4 \times 0.35 - 8 \times 0.36 - 7 \times 0.36 + 2 \times 0.36 = -0.08 \]

Similarly sensitivity to second factor = \(-4.40\)
Using PCA to calculate VaR continued

- As an approximation

\[ \Delta P = -0.08 f_1 - 4.40 f_2 \]

- The \( f_1 \) and \( f_2 \) are independent

- The standard deviation of \( \Delta P \) (from Table 18.4) is

\[ \sqrt{0.08^2 \times 17.49^2 + 4.40^2 \times 6.05^2} = 26.66 \]

- The 1 day 99% VaR is \( 26.66 \times 2.33 = 62.12 \)
Estimating Volatilities and Correlations

Chapter 19
Standard Approach to Estimating Volatility (page 461)

- Define $\sigma_n$ as the volatility per day between day $n-1$ and day $n$, as estimated at end of day $n-1$
- Define $S_i$ as the value of market variable at end of day $i$
- Define $u_i = \ln(S_i/S_{i-1})$

\[
\sigma_n^2 = \frac{1}{m-1} \sum_{i=1}^{m} (u_{n-i} - \bar{u})^2
\]

\[
\bar{u} = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}
\]
Simplifications Usually Made (page 462)

- Define \( u_i \) as \( (S_i - S_{i-1})/S_{i-1} \)
- Assume that the mean value of \( u_i \) is zero
- Replace \( m-1 \) by \( m \)

This gives

\[
\sigma_n^2 = \frac{1}{m} \sum_{i=1}^{m} u_{n-i}^2
\]
Weighting Scheme

Instead of assigning equal weights to the observations we can set

\[ \sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2 \]

where

\[ \sum_{i=1}^{m} \alpha_i = 1 \]
ARCH(m) Model (page 463)

In an ARCH(m) model we also assign some weight to the long-run variance rate, $V_L$:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$
EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the $u^2$ decline exponentially as we move back through time.

- This leads to

$$\sigma_n^2 = \lambda \sigma_{n-1}^2 + (1 - \lambda) u_{n-1}^2$$
Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- RiskMetrics uses $\lambda = 0.94$ for daily volatility forecasting
In GARCH (1,1) we assign some weight to the long-run average variance rate

\[ \sigma_{n}^2 = \gamma V_{L} + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2 \]

Since weights must sum to 1

\[ \gamma + \alpha + \beta = 1 \]
GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

\[
\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2
\]

and

\[
V_L = \frac{\omega}{1 - \alpha - \beta}
\]
Example  (Example 19.2, page 465)

- Suppose

\[ \sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2 \]

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4\%
Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

\[
0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336
\]

The new volatility is 1.53% per day
GARCH \((p,q)\)

\[
\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2
\]
Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring.
Example 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, $p$, that it happens?
- The probability of the event happening on one particular trial and not on the others is $p(1-p)^9$
- We maximize this to obtain a maximum likelihood estimate. Result: $p=0.1$
Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize:

\[
\prod_{i=1}^{m} \left[ \frac{1}{\sqrt{2\pi\nu}} \exp\left( -\frac{u_i^2}{2\nu} \right) \right]
\]

Taking logarithms this is equivalent to maximizing:

\[
\sum_{i=1}^{m} \left[ -\ln(\nu) - \frac{u_i^2}{\nu} \right]
\]

Result:

\[
\nu = \frac{1}{m} \sum_{i=1}^{m} u_i^2
\]
We choose parameters that maximize

\[
\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi \nu_i}} \exp\left(-\frac{u_i^2}{2\nu_i}\right)
\]

or

\[
\sum_{i=1}^{m} \left[-\ln(\nu_i) - \frac{u_i^2}{\nu_i}\right]
\]
Excel Application (Table 19.1, page 469)

- Start with trial values of $\omega$, $\alpha$, and $\beta$
- Update variances
- Calculate
  \[
  \sum_{i=1}^{m} \left[ -\ln(v_i) - \frac{u_i^2}{v_i} \right]
  \]
- Use solver to search for values of $\omega$, $\alpha$, and $\beta$
  that maximize this objective function
- Important note: set up spreadsheet so that you are searching for
  three numbers that are the same order of magnitude (See page 470)
Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting.
- We set the long-run average volatility equal to the sample variance.
- Only two other parameters then have to be estimated.
How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the $u_i^2$ with the autocorrelation of the $u_i^2/\sigma_i^2$
A few lines of algebra shows that

\[ E[\sigma^2_{n+k}] = V_L + (\alpha + \beta)^k (\sigma_n^2 - V_L) \]

The variance rate for an option expiring on day \( m \) is

\[ \frac{1}{m} \sum_{k=0}^{m-1} E[\sigma^2_{n+k}] \]
Forecasting Future Volatility continued (equation 19.4, page 473)

Define

\[ a = \ln \frac{1}{\alpha + \beta} \]

The volatility per annum for a \( T \)-day option is

\[
\sqrt{252 \left( V_L + \frac{1-e^{-aT}}{aT} [V(0) - V_L] \right)}
\]
Volatility Term Structures (Table 19.4)

- The GARCH (1,1) suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities.
- Impact of 1% change in instantaneous volatility for Japanese yen example:

<table>
<thead>
<tr>
<th>Option Life (days)</th>
<th>10</th>
<th>30</th>
<th>50</th>
<th>100</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>Volatility increase (%)</td>
<td>0.84</td>
<td>0.61</td>
<td>0.46</td>
<td>0.27</td>
<td>0.06</td>
</tr>
</tbody>
</table>
Define $x_i = (X_i - X_{i-1}) / X_{i-1}$ and $y_i = (Y_i - Y_{i-1}) / Y_{i-1}$

Also

$\sigma_{x,n}$: daily vol of $X$ calculated on day $n-1$

$\sigma_{y,n}$: daily vol of $Y$ calculated on day $n-1$

$\text{cov}_n$: covariance calculated on day $n-1$

The correlation is $\text{cov}_n / (\sigma_{u,n} \sigma_{v,n})$
Updating Correlations

- We can use similar models to those for volatilities
- Under EWMA

\[ \text{cov}_n = \lambda \text{cov}_{n-1} + (1-\lambda)x_{n-1}y_{n-1} \]
Positive Finite Definite Condition

A variance-covariance matrix, $\Omega$, is internally consistent if the positive semi-definite condition

$$w^T \Omega w \geq 0$$

for all vectors $w$
Example

The variance covariance matrix

\[
\begin{pmatrix}
1 & 0 & 0.9 \\
0 & 1 & 0.9 \\
0.9 & 0.9 & 1
\end{pmatrix}
\]

is not internally consistent
Credit Risk

Chapter 20
Credit Ratings

- In the S&P rating system, AAA is the best rating. After that comes AA, A, BBB, BB, B, and CCC.
- The corresponding Moody’s ratings are Aaa, Aa, A, Baa, Ba, B, and Caa.
- Bonds with ratings of BBB (or Baa) and above are considered to be “investment grade.”
Historical Data

Historical data provided by rating agencies are also used to estimate the probability of default.
Cumulative Ave Default Rates (%)  
(1970-2003, Moody’s, Table 20.1, page 482)

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.04</td>
<td>0.12</td>
<td>0.29</td>
<td>0.62</td>
</tr>
<tr>
<td>Aa</td>
<td>0.02</td>
<td>0.03</td>
<td>0.06</td>
<td>0.15</td>
<td>0.24</td>
<td>0.43</td>
<td>0.68</td>
</tr>
<tr>
<td>A</td>
<td>0.02</td>
<td>0.09</td>
<td>0.23</td>
<td>0.38</td>
<td>0.54</td>
<td>0.91</td>
<td>1.59</td>
</tr>
<tr>
<td>Baa</td>
<td>0.20</td>
<td>0.57</td>
<td>1.03</td>
<td>1.62</td>
<td>2.16</td>
<td>3.24</td>
<td>5.10</td>
</tr>
<tr>
<td>Ba</td>
<td>1.26</td>
<td>3.48</td>
<td>6.00</td>
<td>8.59</td>
<td>11.17</td>
<td>15.44</td>
<td>21.01</td>
</tr>
<tr>
<td>B</td>
<td>6.21</td>
<td>13.76</td>
<td>20.65</td>
<td>26.66</td>
<td>31.99</td>
<td>40.79</td>
<td>50.02</td>
</tr>
<tr>
<td>Caa</td>
<td>23.65</td>
<td>37.20</td>
<td>48.02</td>
<td>55.56</td>
<td>60.83</td>
<td>69.36</td>
<td>77.91</td>
</tr>
</tbody>
</table>
Interpretation

- The table shows the probability of default for companies starting with a particular credit rating.
- A company with an initial credit rating of Baa has a probability of 0.20% of defaulting by the end of the first year, 0.57% by the end of the second year, and so on.
Do Default Probabilities Increase with Time?

- For a company that starts with a good credit rating default probabilities tend to increase with time.
- For a company that starts with a poor credit rating default probabilities tend to decrease with time.
The default intensity (also called hazard rate) is the probability of default for a certain time period conditional on no earlier default.

The unconditional default probability is the probability of default for a certain time period as seen at time zero.

What are the default intensities and unconditional default probabilities for a Caa rate company in the third year?
Recovery Rate

The recovery rate for a bond is usually defined as the price of the bond immediately after default as a percent of its face value.
## Recovery Rates
(Moody’s: 1982 to 2003, Table 20.2, page 483)

<table>
<thead>
<tr>
<th>Class</th>
<th>Mean(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Senior Secured</td>
<td>51.6</td>
</tr>
<tr>
<td>Senior Unsecured</td>
<td>36.1</td>
</tr>
<tr>
<td>Senior Subordinated</td>
<td>32.5</td>
</tr>
<tr>
<td>Subordinated</td>
<td>31.1</td>
</tr>
<tr>
<td>Junior Subordinated</td>
<td>24.5</td>
</tr>
</tbody>
</table>
Estimating Default Probabilities

- Alternatives:
  - Use Bond Prices
  - Use CDS spreads
  - Use Historical Data
  - Use Merton’s Model
Average default intensity over life of bond is approximately
\[ \frac{s}{1 - R} \]

where \( s \) is the spread of the bond’s yield over the risk-free rate and \( R \) is the recovery rate.
More Exact Calculation

- Assume that a five year corporate bond pays a coupon of 6% per annum (semiannually). The yield is 7% with continuous compounding and the yield on a similar risk-free bond is 5% (with continuous compounding).
- Price of risk-free bond is 104.09; price of corporate bond is 95.34; expected loss from defaults is 8.75.
- Suppose that the probability of default is $Q$ per year and that defaults always happen half way through a year (immediately before a coupon payment.)
## Calculations  (Table 20.3, page 485)

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Def Prob</th>
<th>Recovery Amount</th>
<th>Risk-free Value</th>
<th>LGD</th>
<th>Discount Factor</th>
<th>PV of Exp Loss</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>Q</td>
<td>40</td>
<td>106.73</td>
<td>66.73</td>
<td>0.9753</td>
<td>65.08Q</td>
</tr>
<tr>
<td>1.5</td>
<td>Q</td>
<td>40</td>
<td>105.97</td>
<td>65.97</td>
<td>0.9277</td>
<td>61.20Q</td>
</tr>
<tr>
<td>2.5</td>
<td>Q</td>
<td>40</td>
<td>105.17</td>
<td>65.17</td>
<td>0.8825</td>
<td>57.52Q</td>
</tr>
<tr>
<td>3.5</td>
<td>Q</td>
<td>40</td>
<td>104.34</td>
<td>64.34</td>
<td>0.8395</td>
<td>54.01Q</td>
</tr>
<tr>
<td>4.5</td>
<td>Q</td>
<td>40</td>
<td>103.46</td>
<td>63.46</td>
<td>0.7985</td>
<td>50.67Q</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>288.48Q</td>
</tr>
</tbody>
</table>
Calculations continued

- We set $288.48Q = 8.75$ to get $Q = 3.03\%$
- This analysis can be extended to allow defaults to take place more frequently
- With several bonds we can use more parameters to describe the default probability distribution
The Risk-Free Rate

- The risk-free rate when default probabilities are estimated is usually assumed to be the LIBOR/swap zero rate (or sometimes 10 bps below the LIBOR/swap rate)
- To get direct estimates of the spread of bond yields over swap rates we can look at asset swaps
Real World vs Risk-Neutral Default Probabilities

- The default probabilities backed out of bond prices or credit default swap spreads are risk-neutral default probabilities.
- The default probabilities backed out of historical data are real-world default probabilities.
A Comparison

- Calculate 7-year default intensities from the Moody’s data (These are real world default probabilities)
- Use Merrill Lynch data to estimate average 7-year default intensities from bond prices (these are risk-neutral default intensities)
- Assume a risk-free rate equal to the 7-year swap rate minus 10 basis point
# Real World vs Risk Neutral Default Probabilities, 7 year averages

(Table 20.4, page 487)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Real-world default probability per yr (bps)</th>
<th>Risk-neutral default probability per yr (bps)</th>
<th>Ratio</th>
<th>Difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>4</td>
<td>67</td>
<td>16.8</td>
<td>63</td>
</tr>
<tr>
<td>Aa</td>
<td>6</td>
<td>78</td>
<td>13.0</td>
<td>72</td>
</tr>
<tr>
<td>A</td>
<td>13</td>
<td>128</td>
<td>9.8</td>
<td>115</td>
</tr>
<tr>
<td>Baa</td>
<td>47</td>
<td>238</td>
<td>5.1</td>
<td>191</td>
</tr>
<tr>
<td>Ba</td>
<td>240</td>
<td>507</td>
<td>2.1</td>
<td>267</td>
</tr>
<tr>
<td>B</td>
<td>749</td>
<td>902</td>
<td>1.2</td>
<td>153</td>
</tr>
<tr>
<td>Caa</td>
<td>1690</td>
<td>2130</td>
<td>1.3</td>
<td>440</td>
</tr>
</tbody>
</table>
## Risk Premiums Earned By Bond Traders

(Table 20.5, page 488)

<table>
<thead>
<tr>
<th>Rating</th>
<th>Bond Yield Spread over Treasuries (bps)</th>
<th>Spread of risk-free rate used by market over Treasuries (bps)</th>
<th>Spread to compensate for default rate in the real world (bps)</th>
<th>Extra Risk Premium (bps)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaa</td>
<td>83</td>
<td>43</td>
<td>2</td>
<td>38</td>
</tr>
<tr>
<td>Aa</td>
<td>90</td>
<td>43</td>
<td>4</td>
<td>43</td>
</tr>
<tr>
<td>A</td>
<td>120</td>
<td>43</td>
<td>8</td>
<td>69</td>
</tr>
<tr>
<td>Baa</td>
<td>186</td>
<td>43</td>
<td>28</td>
<td>115</td>
</tr>
<tr>
<td>Ba</td>
<td>347</td>
<td>43</td>
<td>144</td>
<td>160</td>
</tr>
<tr>
<td>B</td>
<td>585</td>
<td>43</td>
<td>449</td>
<td>93</td>
</tr>
<tr>
<td>Caa</td>
<td>1321</td>
<td>43</td>
<td>1014</td>
<td>264</td>
</tr>
</tbody>
</table>
Possible Reasons for These Results

- Corporate bonds are relatively illiquid
- The subjective default probabilities of bond traders may be much higher than the estimates from Moody’s historical data
- Bonds do not default independently of each other. This leads to systematic risk that cannot be diversified away.
- Bond returns are highly skewed with limited upside. The non-systematic risk is difficult to diversify away and may be priced by the market
Which World Should We Use?

- We should use risk-neutral estimates for valuing credit derivatives and estimating the present value of the cost of default.
- We should use real world estimates for calculating credit VaR and scenario analysis.
Merton’s Model (page 489-491)

- Merton’s model regards the equity as an option on the assets of the firm.
- In a simple situation the equity value is
  \[
  \max(V_T - D, 0)
  \]
  where \( V_T \) is the value of the firm and \( D \) is the debt repayment required.
Equity vs. Assets

An option pricing model enables the value of the firm’s equity today, $E_0$, to be related to the value of its assets today, $V_0$, and the volatility of its assets, $\sigma_V$

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/D) + (r + \sigma_V^2 / 2)T}{\sigma_V \sqrt{T}}; \quad d_2 = d_1 - \sigma_V \sqrt{T}$$
Volatilities

\[ \sigma_E E_0 = \frac{\partial E}{\partial V} \sigma_V V_0 = N(d_1) \sigma_V V_0 \]

This equation together with the option pricing relationship enables \( V_0 \) and \( \sigma_V \) to be determined from \( E_0 \) and \( \sigma_E \)
Example

- A company’s equity is $3 million and the volatility of the equity is 80%
- The risk-free rate is 5%, the debt is $10 million and time to debt maturity is 1 year
- Solving the two equations yields $V_0 = 12.40$ and $\sigma_v = 21.23\%$
Example continued

- The probability of default is $N(-d_2)$ or 12.7%
- The market value of the debt is 9.40
- The present value of the promised payment is 9.51
- The expected loss is about 1.2%
- The recovery rate is 91%
The Implementation of Merton’s Model (e.g. Moody’s KMV)

- Choose time horizon
- Calculate cumulative obligations to time horizon. This is termed by KMV the “default point”. We denote it by $D$
- Use Merton’s model to calculate a theoretical probability of default
- Use historical data or bond data to develop a one-to-one mapping of theoretical probability into either real-world or risk-neutral probability of default.
Credit Risk in Derivatives Transactions (page 491-493)

Three cases

- Contract always an asset
- Contract always a liability
- Contract can be an asset or a liability
General Result

- Assume that default probability is independent of the value of the derivative.
- Consider times $t_1, t_2, \ldots, t_n$ and default probability is $q_i$ at time $t_i$. The value of the contract at time $t_i$ is $f_i$ and the recovery rate is $R$.
- The loss from defaults at time $t_i$ is $q_i(1-R)E[\max(f_i,0)]$.
- Defining $u_i = q_i(1-R)$ and $v_i$ as the value of a derivative that provides a payoff of $\max(f_i,0)$ at time $t_i$, the cost of defaults is

\[ \sum_{i=1}^{n} u_i v_i \]
Credit Risk Mitigation

- Netting
- Collateralization
- Downgrade triggers
Default Correlation

- The credit default correlation between two companies is a measure of their tendency to default at about the same time.
- Default correlation is important in risk management when analyzing the benefits of credit risk diversification.
- It is also important in the valuation of some credit derivatives, e.g., a first-to-default CDS and CDO tranches.
Measurement

- There is no generally accepted measure of default correlation
- Default correlation is a more complex phenomenon than the correlation between two random variables
Binomial Correlation Measure
(page 499)

- One common default correlation measure, between companies $i$ and $j$ is the correlation between
  - A variable that equals 1 if company $i$ defaults between time 0 and time $T$ and zero otherwise
  - A variable that equals 1 if company $j$ defaults between time 0 and time $T$ and zero otherwise
- The value of this measure depends on $T$. Usually it increases at $T$ increases.
Denote $Q_i(T)$ as the probability that company $A$ will default between time zero and time $T$, and $P_{ij}(T)$ as the probability that both $i$ and $j$ will default. The default correlation measure is

$$\beta_{ij}(T) = \frac{P_{ij}(T) - Q_i(T)Q_j(T)}{\sqrt{[Q_i(T) - Q_i(T)^2][Q_j(T) - Q_j(T)^2]}}$$
Survival Time Correlation

- Define $t_i$ as the time to default for company $i$ and $Q_i(t_i)$ as the probability distribution for $t_i$
- The default correlation between companies $i$ and $j$ can be defined as the correlation between $t_i$ and $t_j$
- But this does not uniquely define the joint probability distribution of default times
Gaussian Copula Model (page 496-499)

- Define a one-to-one correspondence between the time to default, \( t_i \), of company \( i \) and a variable \( x_i \) by
  \[
  Q_i(t_i) = N(x_i) \quad \text{or} \quad x_i = N^{-1}[Q(t_i)]
  \]
  where \( N \) is the cumulative normal distribution function.

- This is a “percentile to percentile” transformation. The \( p \) percentile point of the \( Q_i \) distribution is transformed to the \( p \) percentile point of the \( x_i \) distribution. \( x_i \) has a standard normal distribution.

- We assume that the \( x_i \) are multivariate normal. The default correlation measure, \( \rho_{ij} \) between companies \( i \) and \( j \) is the correlation between \( x_i \) and \( x_j \).
Binomial vs Gaussian Copula Measures (Equation 20.10, page 499)

The measures can be calculated from each other

\[ P_{ij}(T) = M[x_i, x_j; \rho_{ij}] \]

so that

\[ \beta_{ij}(T) = \frac{M[x_i, x_j; \rho_{ij}] - Q_i(T)Q_j(T)}{\sqrt{(Q_i(T) - Q_i(T)^2)(Q_j(T) - Q_j(T)^2)}} \]

where \( M \) is the cumulative bivariate normal probability distribution function
Comparison (Example 20.4, page 499)

- The correlation number depends on the correlation metric used
- Suppose $T = 1$, $Q_i(T) = Q_j(T) = 0.01$, a value of $\rho_{ij}$ equal to 0.2 corresponds to a value of $\beta_{ij}(T)$ equal to 0.024.
- In general $\beta_{ij}(T) < \rho_{ij}$ and $\beta_{ij}(T)$ is an increasing function of $T$
Example of Use of Gaussian Copula
(Example 20.3, page 498)

Suppose that we wish to simulate the defaults for $n$ companies. For each company the cumulative probabilities of default during the next 1, 2, 3, 4, and 5 years are 1%, 3%, 6%, 10%, and 15%, respectively.
Use of Gaussian Copula continued

- We sample from a multivariate normal distribution to get the $x_i$.
- Critical values of $x_i$ are:
  \[ N^{-1}(0.01) = -2.33, \quad N^{-1}(0.03) = -1.88, \]
  \[ N^{-1}(0.06) = -1.55, \quad N^{-1}(0.10) = -1.28, \]
  \[ N^{-1}(0.15) = -1.04 \]
Use of Gaussian Copula continued

- When sample for a company is less than -2.33, the company defaults in the first year
- When sample is between -2.33 and -1.88, the company defaults in the second year
- When sample is between -1.88 and -1.55, the company defaults in the third year
- When sample is between -1.55 and -1.28, the company defaults in the fourth year
- When sample is between -1.28 and -1.04, the company defaults during the fifth year
- When sample is greater than -1.04, there is no default during the first five years
A One-Factor Model for the Correlation Structure (Equation 20.7, page 498)

\[ x_i = a_i M + \sqrt{1 - a_i^2} Z_i \]

- The correlation between \( x_i \) and \( x_j \) is \( a_i a_j \)
- The \( i \)th company defaults by time \( T \) when \( x_i < N^{-1}[Q_i(T)] \)
  or

\[ Z_i < \frac{N^{-1}[Q_i(T) - a_i M]}{\sqrt{1 - a_i^2}} \]

- The probability of this is

\[ Q_i(T|M) = N \left\{ \frac{N^{-1}[Q_i(T)] - a_i M}{\sqrt{1 - a_i^2}} \right\} \]
Credit VaR (page 499-502)

- Can be defined analogously to Market Risk VaR
- A $T$-year credit VaR with an $X\%$ confidence is the loss level that we are $X\%$ confident will not be exceeded over $T$ years
Calculation from a Factor-Based Gaussian Copula Model (equation 20.11, page 500)

- Consider a large portfolio of loans, each of which has a probability of $Q(T)$ of defaulting by time $T$. Suppose that all pairwise copula correlations are $\rho$ so that all $a_i$’s are $\sqrt{\rho}$
- We are $X\%$ certain that $M$ is less than $N^{-1}(1 - X) = -N^{-1}(X)$
- It follows that the VaR is

$$V(X, T) = N\left\{ \frac{N^{-1}[Q(T)] + \sqrt{\rho}N^{-1}(X)}{\sqrt{1 - \rho}} \right\}$$
CreditMetrics (page 500-502)

- Calculates credit VaR by considering possible rating transitions
- A Gaussian copula model is used to define the correlation between the ratings transitions of different companies
Credit Derivatives

- Derivatives where the payoff depends on the credit quality of a company or country
- The market started to grow fast in the late 1990s
- By 2003 notional principal totaled $3 trillion
Credit Default Swaps

- Buyer of the instrument acquires protection from the seller against a default by a particular company or country (the reference entity).
- Example: Buyer pays a premium of 90 bps per year for $100 million of 5-year protection against company X.
- Premium is known as the credit default spread. It is paid for life of contract or until default.
- If there is a default, the buyer has the right to sell bonds with a face value of $100 million issued by company X for $100 million (Several bonds are typically deliverable).
**CDS Structure** (Figure 21.1, page 508)

Recovery rate, $R$, is the ratio of the value of the bond issued by reference entity immediately after default to the face value of the bond.
Other Details

- Payments are usually made quarterly or semiannually in arrears.
- In the event of default there is a final accrual payment by the buyer.
- Settlement can be specified as delivery of the bonds or in cash.
- Suppose payments are made quarterly in the example just considered. What are the cash flows if there is a default after 3 years and 1 month and recovery rate is 40%?
Attractions of the CDS Market

- Allows credit risks to be traded in the same way as market risks
- Can be used to transfer credit risks to a third party
- Can be used to diversify credit risks
Using a CDS to Hedge a Bond

Portfolio consisting of a 5-year par yield corporate bond that provides a yield of 6% and a long position in a 5-year CDS costing 100 basis points per year is (approximately) a long position in a riskless instrument paying 5% per year.
Valuation Example (page 510-512)

- Conditional on no earlier default a reference entity has a (risk-neutral) probability of default of 2% in each of the next 5 years. (This is a default intensity)
- Assume payments are made annually in arrears, that defaults always happen half way through a year, and that the expected recovery rate is 40%
- Suppose that the breakeven CDS rate is $s$ per dollar of notional principal
# Unconditional Default and Survival Probabilities

(Table 21.1)

<table>
<thead>
<tr>
<th>Time (years)</th>
<th>Default Probability</th>
<th>Survival Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0200</td>
<td>0.9800</td>
</tr>
<tr>
<td>2</td>
<td>0.0196</td>
<td>0.9604</td>
</tr>
<tr>
<td>3</td>
<td>0.0192</td>
<td>0.9412</td>
</tr>
<tr>
<td>4</td>
<td>0.0188</td>
<td>0.9224</td>
</tr>
<tr>
<td>5</td>
<td>0.0184</td>
<td>0.9039</td>
</tr>
</tbody>
</table>
### Calculation of PV of Payments

Table 21.2 (Principal=$1)

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Survival Prob</th>
<th>Expected Paymt</th>
<th>Discount Factor</th>
<th>PV of Exp Pmt</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9800</td>
<td>0.9800&lt;sub&gt;S&lt;/sub&gt;</td>
<td>0.9512</td>
<td>0.9322&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
<tr>
<td>2</td>
<td>0.9604</td>
<td>0.9604&lt;sub&gt;S&lt;/sub&gt;</td>
<td>0.9048</td>
<td>0.8690&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
<tr>
<td>3</td>
<td>0.9412</td>
<td>0.9412&lt;sub&gt;S&lt;/sub&gt;</td>
<td>0.8607</td>
<td>0.8101&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
<tr>
<td>4</td>
<td>0.9224</td>
<td>0.9224&lt;sub&gt;S&lt;/sub&gt;</td>
<td>0.8187</td>
<td>0.7552&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
<tr>
<td>5</td>
<td>0.9039</td>
<td>0.9039&lt;sub&gt;S&lt;/sub&gt;</td>
<td>0.7788</td>
<td>0.7040&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>4.0704&lt;sub&gt;S&lt;/sub&gt;</td>
</tr>
</tbody>
</table>
## Present Value of Expected Payoff

*(Table 21.3; Principal = $1)*

<table>
<thead>
<tr>
<th>Time (yrs)</th>
<th>Default Probab.</th>
<th>Rec. Rate</th>
<th>Expected Payoff</th>
<th>Discount Factor</th>
<th>PV of Exp. Payoff</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0200</td>
<td>0.4</td>
<td>0.0120</td>
<td>0.9753</td>
<td>0.0117</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0196</td>
<td>0.4</td>
<td>0.0118</td>
<td>0.9277</td>
<td>0.0109</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0192</td>
<td>0.4</td>
<td>0.0115</td>
<td>0.8825</td>
<td>0.0102</td>
</tr>
<tr>
<td>3.5</td>
<td>0.0188</td>
<td>0.4</td>
<td>0.0113</td>
<td>0.8395</td>
<td>0.0095</td>
</tr>
<tr>
<td>4.5</td>
<td>0.0184</td>
<td>0.4</td>
<td>0.0111</td>
<td>0.7985</td>
<td>0.0088</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td>0.0111</td>
<td></td>
<td>0.0511</td>
</tr>
</tbody>
</table>
PV of Accrual Payment Made in Event of a Default. (Table 21.4; Principal=$1)

<table>
<thead>
<tr>
<th>Time</th>
<th>Default Prob</th>
<th>Expected Accr Pmt</th>
<th>Disc Factor</th>
<th>PV of Pmt</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0.0200</td>
<td>0.0100\textsubscript{$s$}</td>
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<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
<td>0.0426\textsubscript{$s$}</td>
</tr>
</tbody>
</table>
Putting it all together

- PV of expected payments is $4.0704_s + 0.0426_s = 4.1130_s$
- The breakeven CDS spread is given by $4.1130_s = 0.0511$ or $s = 0.0124$ (124 bps)
- The value of a swap negotiated some time ago with a CDS spread of 150 bps would be $4.1130 \times 0.0150 - 0.0511$ or 0.0106 times the principal.
Implying Default Probabilities from CDS spreads

- Suppose that the mid market spread for a 5 year newly issued CDS is 100bps per year.
- We can reverse engineer our calculations to conclude that the default intensity is 1.61% per year.
- If probabilities are implied from CDS spreads and then used to value another CDS the result is not sensitive to the recovery rate providing the same recovery rate is used throughout.
Other Credit Derivatives

- Binary CDS
- First-to-default Basket CDS
- Total return swap
- Credit default option
- Collateralized debt obligation
Binary CDS (page 513)

- The payoff in the event of default is a fixed cash amount
- In our example the PV of the expected payoff for a binary swap is 0.0852 and the breakeven binary CDS spread is 207 bps
Example: European option to buy 5 year protection on Ford for 280 bps starting in one year. If Ford defaults during the one-year life of the option, the option is knocked out.

- Depends on the volatility of CDS spreads.
Total Return Swap (page 515-516)

- Agreement to exchange total return on a corporate bond for LIBOR plus a spread
- At the end there is a payment reflecting the change in value of the bond
- Usually used as financing tools by companies that want an investment in the corporate bond
First to Default Basket CDS
(page 516)

- Similar to a regular CDS except that several reference entities are specified and there is a payoff when the first one defaults
- This depends on “default correlation”
- Second, third, and $n$th to default deals are defined similarly
Collateralized Debt Obligation
(Figure 21.3, page 517)

- A pool of debt issues are put into a special purpose trust
- Trust issues claims against the debt in a number of tranches
  - First tranche covers $x\%$ of notional and absorbs first $x\%$ of default losses
  - Second tranche covers $y\%$ of notional and absorbs next $y\%$ of default losses
  - etc
- A tranche earn a promised yield on remaining principal in the tranche
CDO Structure

- Bond 1
- Bond 2
- Bond 3
- Bond n

Average Yield 8.5%

Trust

- Tranche 1
  - 1st 5% of loss
  - Yield = 35%

- Tranche 2
  - 2nd 10% of loss
  - Yield = 15%

- Tranche 3
  - 3rd 10% of loss
  - Yield = 7.5%

- Tranche 4
  - Residual loss
  - Yield = 6%
Synthetic CDO

Instead of buying the bonds the arranger of the CDO sells credit default swaps.
Single Tranche Trading (Table 21.6, page 518)

- This involves trading tranches of standard portfolios that are not funded
- CDX IG (Aug 4, 2004):

<table>
<thead>
<tr>
<th>Tranche</th>
<th>0-3%</th>
<th>3-7%</th>
<th>7-10%</th>
<th>10-15%</th>
<th>15-30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quote</td>
<td>41.8%</td>
<td>347bps</td>
<td>135.5bps</td>
<td>47.5bps</td>
<td>14.5bps</td>
</tr>
</tbody>
</table>

- iTraxx IG (Aug 4, 2004)

<table>
<thead>
<tr>
<th>Tranche</th>
<th>0-3%</th>
<th>3-6%</th>
<th>6-9%</th>
<th>9-12%</th>
<th>12-22%</th>
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</thead>
<tbody>
<tr>
<td>Quote</td>
<td>27.6%</td>
<td>168bps</td>
<td>70bps</td>
<td>43bps</td>
<td>20bps</td>
</tr>
</tbody>
</table>
Valuation of Correlation Dependent Credit Derivatives (page 519-520)

- A popular approach is to use a factor-based Gaussian copula model to define correlations between times to default the time to default.
- Often all pairwise correlations and all the unconditional default distributions are assumed to be the same.
- Market likes to imply a pairwise correlation from market quotes.
Valuation of Correlation Dependent Credit Derivatives continued

\[ Q(T|M) = N \left( \frac{N^{-1}[Q(T)] - \sqrt{\rho M}}{\sqrt{1-\rho}} \right) \]

- The probability of \( k \) defaults by time \( T \) conditional on \( M \) is

\[ \frac{N!}{(N-k)!k!} \left( Q(T|M)^{k} \left[ 1 - Q(T|M) \right]^{N-k} \right) \]

- This enables cash flows conditional on \( M \) to be calculated. By integrating over \( M \) the unconditional distributions are obtained.
Convertible Bonds

- Often valued with a tree where during a time interval $\Delta t$ there is
  - a probability $p_u$ of an up movement
  - A probability $p_d$ of a down movement
  - A probability $1 - \exp(-\lambda t)$ that there will be a default

- In the event of a default the stock price falls to zero and there is a recovery on the bond
The Probabilities

\[ p_u = \frac{a - de^{-\lambda \Delta t}}{u - d} \]
\[ p_d = \frac{ue^{-\lambda \Delta t} - a}{u - d} \]
\[ u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}} \]
\[ d = \frac{1}{u} \]
Node Calculations

Define:

$Q_1$: value of bond if neither converted nor called

$Q_2$: value of bond if called

$Q_3$: value of bond if converted

Value at a node $= \max[\min(Q_1, Q_2), Q_3]$
Example 21.1  (page 522)

- 9-month zero-coupon bond with face value of $100
- Convertible into 2 shares
- Callable for $113 at any time
- Initial stock price = $50,
- volatility = 30%,
- no dividends
- Risk-free rates all 5%
- Default intensity, \( \lambda \), is 1%
- Recovery rate=40%
The Tree (Figure 21.4, page 522)
Exotic Options

Chapter 22
Types of Exotics

- Package
- Nonstandard American options
- Forward start options
- Compound options
- Chooser options
- Barrier options
- Binary options
- Lookback options
- Shout options
- Asian options
- Options to exchange one asset for another
- Options involving several assets
Packages (page 529)

- Portfolios of standard options
- Examples from Chapter 10: bull spreads, bear spreads, straddles, etc
- Often structured to have zero cost
- One popular package is a range forward contract
Non-Standard American Options
(page 530)

- Exercisable only on specific dates (Bermudans)
- Early exercise allowed during only part of life (initial “lock out” period)
- Strike price changes over the life (warrants, convertibles)
Forward Start Options (page 531)

- Option starts at a future time, $T_1$
- Most common in employee stock option plans
- Often structured so that strike price equals asset price at time $T_1$
Compound Option (page 531)

- Option to buy or sell an option
  - Call on call
  - Put on call
  - Call on put
  - Put on put
- Can be valued analytically
- Price is quite low compared with a regular option
Chooser Option “As You Like It”
(page 532)

- Option starts at time 0, matures at $T_2$
- At $T_1$ ($0 < T_1 < T_2$) buyer chooses whether it is a put or call
- This is a package!
Chooser Option as a Package

At time $T_1$ the value is $\max(c, p)$

From put-call parity

$$p = c + e^{-r(T_2-T_1)}K - S_1e^{-q(T_2-T_1)}$$

The value at time $T_1$ is therefore

$$c + e^{-q(T_2-T_1)}\max(0, Ke^{-(r-q)(T_2-T_1)} - S_1)$$

This is a call maturing at time $T_2$ plus a put maturing at time $T_1$
Barrier Options  (page 535)

- Option comes into existence only if stock price hits barrier before option maturity
  - ‘In’ options

- Option dies if stock price hits barrier before option maturity
  - ‘Out’ options
Barrier Options (continued)

- Stock price must hit barrier from below
  - ‘Up’ options
- Stock price must hit barrier from above
  - ‘Down’ options
- Option may be a put or a call
- Eight possible combinations
Parity Relations

\[ c = c_{ui} + c_{uo} \]
\[ c = c_{di} + c_{do} \]
\[ p = p_{ui} + p_{uo} \]
\[ p = p_{di} + p_{do} \]
Binary Options  (page 535)

- **Cash-or-nothing**: pays $Q$ if $S_T > K$, otherwise pays nothing.
  - Value = $e^{-rT} Q \, N(d_2)$

- **Asset-or-nothing**: pays $S_T$ if $S_T > K$, otherwise pays nothing.
  - Value = $S_0 \, N(d_1)$
Decomposition of a Call Option

Long Asset-or-Nothing option
Short Cash-or-Nothing option where payoff is $K$

Value = $S_0 \, N(d_1) - e^{-rT} \, KN(d_2)$
Lookback Options  (page 536)

- Lookback call pays $S_T - S_{\text{min}}$ at time $T$
- Allows buyer to buy stock at lowest observed price in some interval of time
- Lookback put pays $S_{\text{max}} - S_T$ at time $T$
- Allows buyer to sell stock at highest observed price in some interval of time
- Analytic solution
Shout Options

- Buyer can ‘shout’ once during option life
- Final payoff is either
  - Usual option payoff, $\max(S_T - K, 0)$, or
  - Intrinsic value at time of shout, $S_{\tau} - K$
- Payoff: $\max(S_T - S_{\tau}, 0) + S_{\tau} - K$
- Similar to lookback option but cheaper
- How can a binomial tree be used to value a shout option?
Asian Options (page 538)

- Payoff related to average stock price
- Average Price options pay:
  - Call: $\max(S_{\text{ave}} - K, 0)$
  - Put: $\max(K - S_{\text{ave}}, 0)$
- Average Strike options pay:
  - Call: $\max(S_T - S_{\text{ave}}, 0)$
  - Put: $\max(S_{\text{ave}} - S_T, 0)$
Asian Options

- No analytic solution
- Can be valued by assuming (as an approximation) that the average stock price is lognormally distributed
Exchange Options  (page 540)

- Option to exchange one asset for another
- For example, an option to exchange one unit of $U$ for one unit of $V$
- Payoff is $\max(V_T - U_T, 0)$
Basket Options (page 541)

- A basket option is an option to buy or sell a portfolio of assets
- This can be valued by calculating the first two moments of the value of the basket and then assuming it is lognormal
How Difficult is it to Hedge Exotic Options?

- In some cases exotic options are easier to hedge than the corresponding vanilla options. (e.g., Asian options)
- In other cases they are more difficult to hedge (e.g., barrier options)
Static Options Replication
(Section 22.13, page 541)

- This involves approximately replicating an exotic option with a portfolio of vanilla options.
- Underlying principle: if we match the value of an exotic option on some boundary, we have matched it at all interior points of the boundary.
- Static options replication can be contrasted with dynamic options replication where we have to trade continuously to match the option.
Example

- A 9-month up-and-out call option on a non-dividend paying stock where $S_0 = 50$, $K = 50$, the barrier is 60, $r = 10\%$, and $\sigma = 30\%$
- Any boundary can be chosen but the natural one is
  
  \[ c(S, 0.75) = \max(S - 50, 0) \text{ when } S < 60 \]
  
  \[ c(60, t ) = 0 \text{ when } 0 \leq t \leq 0.75 \]
Example (continued)

- We might try to match the following points on the boundary

\[ c(S, 0.75) = \text{MAX}(S - 50, 0) \] for \( S < 60 \)

\[ c(60, 0.50) = 0 \]

\[ c(60, 0.25) = 0 \]

\[ c(60, 0.00) = 0 \]
Example continued
(See Table 22.1, page 543)

We can do this as follows:
+1.00 call with maturity 0.75 & strike 50
−2.66 call with maturity 0.75 & strike 60
+0.97 call with maturity 0.50 & strike 60
+0.28 call with maturity 0.25 & strike 60
Example (continued)

- This portfolio is worth 0.73 at time zero compared with 0.31 for the up-and-out option.
- As we use more options, the value of the replicating portfolio converges to the value of the exotic option.
- For example, with 18 points matched on the horizontal boundary the value of the replicating portfolio reduces to 0.38; with 100 points being matched it reduces to 0.32.
Using Static Options Replication

- To hedge an exotic option we short the portfolio that replicates the boundary conditions.
- The portfolio must be unwound when any part of the boundary is reached.
Weather, Energy, and Insurance Derivatives

Chapter 23
Pricing Issues  (page 551)

- To a good approximation many underlying variables in insurance, weather, and energy derivatives contracts can be assumed to have zero systematic risk.
- This means that we can calculate expected payoff in the real world and discount at the risk-free rate.
Weather Derivatives: Definitions
(page 552)

- Heating degree days (HDD): For each day this is \( \max(0, 65 - A) \) where \( A \) is the average of the highest and lowest temperature in °F.
- Cooling Degree Days (CDD): For each day this is \( \max(0, A - 65) \)
- Contracts specify the weather station to be used
Weather Derivatives: Products

- A typical product is a forward contract or an option on the cumulative CDD or HDD during a month.
- Weather derivatives are often used by energy companies to hedge the volume of energy required for heating or cooling during a particular month.
- How would you value an option on August CDD at a particular weather station?
Energy Derivatives (page 553-556)

Main energy sources:
- Oil
- Gas
- Electricity
Oil Derivatives (page 553)

- Virtually all derivatives available on stocks and stock indices are also available in the OTC market with oil as the underlying asset.
- Futures and futures options traded on the New York Mercantile Exchange (NYMEX) and the International Petroleum Exchange (IPE) are also popular.
A typical OTC contract is for the delivery of a specified amount of natural gas at a roughly uniform rate to specified location during a month.

NYMEX and IPE trade contracts that require delivery of 10,000 million British thermal units of natural gas to a specified location.
Electricity Derivatives

- Electricity is an unusual commodity in that it cannot be stored.
- The U.S is divided into about 140 control areas and a market for electricity is created by trading between control areas.
Electricity Derivatives continued

- A typical contract allows one side to receive a specified number of megawatt hours for a specified price at a specified location during a particular month
- Types of contracts:
  - $5x8$, $5x16$, $7x24$, daily or monthly exercise, swing options
How an Energy Producer Hedges Risks

- Estimate a relationship of the form
  \[ Y = a + bP + cT + \varepsilon \]
  where \( Y \) is the monthly profit, \( P \) is the average energy prices, \( T \) is temperature, and \( \varepsilon \) is an error term.

- Take a position of \(-b\) in energy forwards and \(-c\) in weather forwards.
Modeling Energy Prices (equation 23.1, page 555)

\[ d \ln S = [\theta(t) - \alpha \ln S] dt + \sigma \, dz \]

- For oil \( \alpha \) is about 0.5 and \( \sigma \) is about 20%; for natural gas these parameters are about 1.0 and 40%; for electricity they are about 15 and 150%. 
Insurance Derivatives (page 556-557)

- CAT bonds are an alternative to traditional reinsurance.
- This is a bond issued by a subsidiary of an insurance company that pays a higher-than-normal interest rate.
- If claims of a certain type are above a certain level the interest and possibly the principal on the bond are used to meet claims.
More on Models and Numerical Procedures

Chapter 24
Three Alternatives to Geometric Brownian Motion

- Constant elasticity of variance (CEV)
- Mixed Jump diffusion
- Variance Gamma
CEV Model (page 562 to 563))

\[ dS = (r - q)Sdt + \sigma S^\alpha dz \]

- When \( \alpha = 1 \) the model is Black-Scholes case
- When \( \alpha > 1 \) volatility rises as stock price rises
- When \( \alpha < 1 \) volatility falls as stock price rises

European option can be value analytically in terms of the cumulative non-central chi square distribution
CEV Models Implied Volatilities

\[ \sigma_{\text{imp}} \]

\[ \alpha < 1 \]

\[ \alpha > 1 \]
Merton produced a pricing formula when the stock price follows a diffusion process overlaid with random jumps

\[ ds / S = (\mu - \lambda k)dt + \sigma dz + dp \]

- \( dp \) is the random jump
- \( k \) is the expected size of the jump
- \( \lambda \) is the probability that a jump occurs in the next interval of length \( dt \)
Jumps and the Smile

- Jumps have a big effect on the implied volatility of short term options
- They have a much smaller effect on the implied volatility of long term options
The Variance-Gamma Model (page 564 to 566)

- $g$ is change over time $T$ in a variable that follows a gamma process. This is a process where small jumps occur frequently and there are occasional large jumps.
- Conditional on $g$, $\ln S_T$ is normal. Its variance proportional to $g$.
- There are 3 parameters:
  - $\nu$, the variance rate of the gamma process
  - $\sigma^2$, the average variance rate of $\ln S$ per unit time
  - $\theta$, a parameter defining skewness
Understanding the Variance-Gamma Model

- $g$ defines the rate at which information arrives during time $T$ ($g$ is sometimes referred to as measuring economic time)
- If $g$ is large the the change in $\ln S$ has a relatively large mean and variance
- If $g$ is small relatively little information arrives and the change in $\ln S$ has a relatively small mean and variance
Time Varying Volatility

- Suppose the volatility is $\sigma_1$ for the first year and $\sigma_2$ for the second and third.
- Total accumulated variance at the end of three years is $\sigma_1^2 + 2\sigma_2^2$.
- The 3-year average volatility is

\[
3\overline{\sigma}^2 = \sigma_1^2 + 2\sigma_2^2; \quad \overline{\sigma} = \sqrt{\frac{\sigma_1^2 + 2\sigma_2^2}{3}}
\]
Stochastic Volatility Models (equations 24.2 and 24.3, page 567)

\[
\frac{dS}{S} = (r-q)dt + \sqrt{V} \, dz_S
\]

\[
dV = \alpha(V_L - V)dt + \xi V^\alpha \, dz_V
\]

- When \( V \) and \( S \) are uncorrelated a European option price is the Black-Scholes price integrated over the distribution of the average variance.
Stochastic Volatility Models continued

- When $V$ and $S$ are negatively correlated we obtain a downward sloping volatility skew similar to that observed in the market for equities.
- When $V$ and $S$ are positively correlated the skew is upward sloping.
The implied volatility function model is designed to create a process for the asset price that exactly matches observed option prices. The usual geometric Brownian motion model
\[ dS = (r - q)Sdt + \sigma Sdz \]
is replaced by
\[ dS = [r(t) - q(t)]dt + \sigma(S, t)Sdz \]
The volatility function that leads to the model matching all European option prices is

\[
[\sigma(K, t)]^2 = 2 \frac{\partial c_{mkt}/\partial t + q(t)c_{mkt} + K[r(t) - q(t)]\partial c_{mkt}/\partial K}{K^2(\partial^2 c_{mkt}/\partial K^2)}
\]
Strengths and Weaknesses of the IVF Model

- The model matches the probability distribution of stock prices assumed by the market at each future time.
- The models does not necessarily get the joint probability distribution of stock prices at two or more times correct.
Numerical Procedures

Topics:
- Path dependent options using tree
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo
Path Dependence: The Traditional View

- Backwards induction works well for American options. It cannot be used for path-dependent options.
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options.
Extension of Backwards Induction

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options
Consider an American lookback put on a stock where $S = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month & the life of the option is 3 months.

Payoff is $S_{\text{max}} - S_T$.

We can value the deal by considering all possible values of the maximum stock price at each node.

(This example is presented to illustrate the methodology. It is not the most efficient way of handling American lookbacks (See Technical Note 13).)
Example: An American Lookback Put Option (Figure 24.2, page 570)

\[ S_0 = 50, \sigma = 40\%, r = 10\%, \Delta t = 1 \text{ month}, \]
Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a “path function”)
- The value of the path function at a node can be calculated from the stock price at the node & from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree
Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node.
- The basic idea is that it is not necessary to consider every possible value of the path function.
- It is sufficient to consider a relatively small number of representative values of the function at each node.
Working Forward

- First work forward through the tree calculating the max and min values of the “path function” at each node.
- Next choose representative values of the path function that span the range between the min and the max.
- Simplest approach: choose the min, the max, and $N$ equally spaced values between the min and max.
Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node.
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation.
Part of Tree to Calculate Value of an Option on the Arithmetic Average
(Figure 24.3, page 572)

S = 50.00

<table>
<thead>
<tr>
<th>Average S</th>
<th>Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>46.65</td>
<td>5.642</td>
</tr>
<tr>
<td>49.04</td>
<td>5.923</td>
</tr>
<tr>
<td>51.44</td>
<td>6.206</td>
</tr>
<tr>
<td>53.83</td>
<td>6.492</td>
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</tbody>
</table>

S = 45.72

<table>
<thead>
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<th>Average S</th>
<th>Option Price</th>
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</thead>
<tbody>
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<td>43.88</td>
<td>3.430</td>
</tr>
<tr>
<td>46.75</td>
<td>3.750</td>
</tr>
<tr>
<td>49.61</td>
<td>4.079</td>
</tr>
<tr>
<td>52.48</td>
<td>4.416</td>
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</tbody>
</table>

S = 54.68

<table>
<thead>
<tr>
<th>Average S</th>
<th>Option Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>47.99</td>
<td>7.575</td>
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<tr>
<td>51.12</td>
<td>8.101</td>
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<tr>
<td>54.26</td>
<td>8.635</td>
</tr>
<tr>
<td>57.39</td>
<td>9.178</td>
</tr>
</tbody>
</table>

S=50, X=50, σ=40%, r=10%, T=1yr, Δt=0.05yr. We are at time 4Δt
Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node X when the average of 5 observations is 51.44

Node Y: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node Z: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

Node X: value is

\[(0.5056 \times 8.247 + 0.4944 \times 4.182)e^{-0.1 \times 0.05} = 6.206\]
Using Trees with Barriers
(Section 24.5, page 573)

- When trees are used to value options with barriers, convergence tends to be slow.
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree.
True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case
Inner and Outer Barriers for Trinomial Trees
(Figure 24.4, page 574)
Alternative Solutions to Valuing Barrier Options

- Interpolate between value when inner barrier is assumed and value when outer barrier is assumed
- Ensure that nodes always lie on the barriers
- Use adaptive mesh methodology

In all cases a trinomial tree is preferable to a binomial tree
Modeling Two Correlated Variables  (Section 24.6, page 576)

APPROACHES:

1. Transform variables so that they are not correlated & build the tree in the transformed variables
2. Take the correlation into account by adjusting the position of the nodes
3. Take the correlation into account by adjusting the probabilities
Monte Carlo Simulation and American Options

- Two approaches:
  - The least squares approach
  - The exercise boundary parameterization approach

- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income
### Sampled Paths

<table>
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<tr>
<th>Path</th>
<th>$t=0$</th>
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<th>$t=3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.00</td>
<td>1.09</td>
<td>1.08</td>
<td>1.34</td>
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<tr>
<td>2</td>
<td>1.00</td>
<td>1.16</td>
<td>1.26</td>
<td>1.54</td>
</tr>
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<td>3</td>
<td>1.00</td>
<td>1.22</td>
<td>1.07</td>
<td>1.03</td>
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<td>0.77</td>
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<td>0.92</td>
<td>0.84</td>
<td>1.01</td>
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<tr>
<td>8</td>
<td>1.00</td>
<td>0.88</td>
<td>1.22</td>
<td>1.34</td>
</tr>
</tbody>
</table>
The Least Squares Approach (page 579)

- We work back from the end using a least squares approach to calculate the continuation value at each time.

- Consider year 2. The option is in the money for five paths. These give observations on $S$ of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00, $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$.
The Least Squares Approach

continued

- Fitting a model of the form $V = a + bS + cS^2$ we get a best fit relation

$$V = -1.070 + 2.983S - 1.813S^2$$

for the continuation value $V$

- This defines the early exercise decision at $t = 2$. We carry out a similar analysis at $t = 1$
The Least Squares Approach

In practice more complex functional forms can be used for the continuation value and many more paths are sampled.
The Early Exercise Boundary Parametrization Approach (page 582)

- We assume that the early exercise boundary can be parameterized in some way.
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values.
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.
Application to Example

● We parameterize the early exercise boundary by specifying a critical asset price, $S^*$, below which the option is exercised.

● At $t=1$ the optimal $S^*$ for the eight paths is 0.88. At $t=2$ the optimal $S^*$ is 0.84.

● In practice we would use many more paths to calculate the $S^*$.
Martingales and Measures

Chapter 25
Derivatives Dependent on a Single Underlying Variable

Consider a variable, θ, (not necessarily the price of a traded security) that follows the process

\[ \frac{d \theta}{\theta} = m \, dt + s \, dz \]

Imagine two derivatives dependent on θ with prices \( f_1 \) and \( f_2 \). Suppose

\[ \frac{d f_1}{f_1} = \mu_1 \, dt + \sigma_1 \, dz \]

\[ \frac{d f_2}{f_2} = \mu_2 \, dt + \sigma_2 \, dz \]
Forming a Riskless Portfolio

We can set up a riskless portfolio $\Pi$, consisting of

\[ + \sigma_2 f_2 \text{ of the 1st derivative and} \]
\[ - \sigma_1 f_1 \text{ of the 2nd derivative} \]

\[ \Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2 \]

\[ \Delta \Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2) \Delta t \]
Market Price of Risk (Page 590)

Since the portfolio is riskless: \( \Delta \Pi = r \Pi \Delta t \)

This gives: \( \mu_1 \sigma_2 - \mu_2 \sigma_1 = r \sigma_2 - r \sigma_1 \)

or \( \frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2} \)

- This shows that \((\mu - r)/\sigma\) is the same for all derivatives dependent on the same underlying variable, \(\theta\)
- We refer to \((\mu - r)/\sigma\) as the market price of risk for \(\theta\) and denote it by \(\lambda\)
Extension of the Analysis
to Several Underlying Variables
(Equations 25.12 and 25.13, page 593)

If $f$ depends on several underlying variables with

$$\frac{df}{f} = \mu \, dt + \sum_{i=1}^{n} \sigma_i \, dz_i$$

then

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$
Martingales (Page 594)

- A martingale is a stochastic process with zero drift
- A variable following a martingale has the property that its expected future value equals its value today
Alternative Worlds

In the traditional risk-neutral world

\[ df = rf \, dt + \sigma f \, dz \]

In a world where the market price of risk is \( \lambda \)

\[ df = (r + \lambda \sigma) \, f \, dt + \sigma f \, dz \]
The Equivalent Martingale Measure Result (Page 595)

If we set $\lambda$ equal to the volatility of a security $g$, then Ito's lemma shows that $f/g$ is a martingale for all derivative security prices $f$ ($f$ and $g$ are assumed to provide no income during the period under consideration).
Forward Risk Neutrality

We refer to a world where the market price of risk is the volatility of \( g \) as a world that is forward risk neutral with respect to \( g \).

If \( E_g \) denotes a world that is FRN wrt \( g \)

\[
\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)
\]
Alternative Choices for the Numeraire Security $g$

- Money Market Account
- Zero-coupon bond price
- Annuity factor
Money Market Account as the Numeraire

- The money market account is an account that starts at $1 and is always invested at the short-term risk-free interest rate.
- The process for the value of the account is
  \[ dg = r_g \, dt \]
- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world.
Money Market Account continued

Since \( g_0 = 1 \) and \( g_T = e^{\int_0^T r dt} \), the equation

\[
\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)
\]

becomes

\[
f_0 = \hat{E} \left[ e^{-\int_0^T r dt} f_T \right]
\]

where \( \hat{E} \) denotes expectations in the traditional risk-neutral world.
Zero-Coupon Bond Maturing at time $T$ as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right)$$

becomes

$$f_0 = P(0,T)E_T[f_T]$$

where $P(0,T)$ is the zero-coupon bond price and $E_T$ denotes expectations in a world that is FRN wrt the bond price
Forward Prices

In a world that is FRN wrt $P(0,T)$, the expected value of a security at time $T$ is its forward price
Interest Rates

In a world that is FRN wrt \( P(0,T_2) \) the expected value of an interest rate lasting between times \( T_1 \) and \( T_2 \) is the forward interest rate.
Annuity Factor as the Numeraire

The equation

\[ \frac{f_0}{g_0} = E_g \left( \frac{f_T}{g_T} \right) \]

becomes

\[ f_0 = A(0) E_A \left[ \frac{f_T}{A(T)} \right] \]
Suppose that $s(t)$ is the swap rate corresponding to the annuity factor $A$. Then:

$$s(t) = E_A[s(T)]$$
Extension to Several Independent Factors (Page 599)

In the traditional risk-neutral world

\[
df(t) = r(t) f(t) dt + \sum_{i=1}^{m} \sigma_{f,i}(t) f(t) dz_i
\]

\[
dg(t) = r(t) g(t) dt + \sum_{i=1}^{m} \sigma_{g,i}(t) g(t) dz_i
\]

For other worlds that are internally consistent

\[
df(t) = \left[ r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{f,i}(t) \right] f(t) dt + \sum_{i=1}^{m} \sigma_{f,i}(t) f(t) dz_i
\]

\[
dg(t) = \left[ r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{g,i}(t) \right] g(t) dt + \sum_{i=1}^{m} \sigma_{g,i}(t) g(t) dz_i
\]
We define a world that is FRN \text{ wrt } g as world where \( \lambda_i = \sigma_{g,i} \)

As in the one-factor case, \( f/g \) is a martingale and the rest of the results hold.
Applications
(Section 25.6, page 600)

- Valuation of a European call option when interest rates are stochastic
- Valuation of an option to exchange one asset for another
When we change the numeraire security from $g$ to $h$, the drift of a variable $v$ increases by $\rho \sigma_v \sigma_q$ where $\sigma_v$ is the volatility of $v$, $w = h/g$, $\sigma_q$ is the volatility of $w$, and $\rho$ is the correlation between $v$ and $w$. 
Chapter 26

Interest Rate Derivatives: The Standard Market Models
The Complications in Valuing Interest Rate Derivatives (page 611)

- We need a whole term structure to define the level of interest rates at any time
- The stochastic process for an interest rate is more complicated than that for a stock price
- Volatilities of different points on the term structure are different
- Interest rates are used for discounting the payoff as well as for defining the payoff
Approaches to Pricing Interest Rate Options

- Use a variant of Black's model
- Use a no-arbitrage (yield curve based) model
Black’s Model

- Similar to the model proposed by Fischer Black for valuing options on futures
- Assumes that the value of an interest rate, a bond price, or some other variable at a particular time \( T \) in the future has a lognormal distribution
Black’s Model (continued)

- The mean of the probability distribution is the forward value of the variable
- The standard deviation of the probability distribution of the log of the variable is \( \sigma \sqrt{T} \)
  where \( \sigma \) is the volatility
- The expected payoff is discounted at the \( T \)-maturity rate observed today
Black’s Model (Eqn 26.1 and 26.2, page 611-612)

\[ c = P(0, T)[F_0 N(d_1) - KN(d_2)] \]
\[ p = P(0, T)[KN(-d_2) - F_0 N(-d_1)] \]
\[ d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \sigma \sqrt{T} \]

- \( K \): strike price
- \( F_0 \): forward value of variable today
- \( T \): option maturity
- \( \sigma \): volatility of \( F \)
Black’s Model: Delayed Payoff

\[ c = P(0, T^*)[F_0N(d_1) - KN(d_2)] \]
\[ p = P(0, T^*)[KN(-d_2) - F_0N(-d_1)] \]
\[ d_1 = \frac{\ln(F_0/K) + \sigma^2 T / 2}{\sigma \sqrt{T}}; \quad d_2 = d_1 - \sigma \sqrt{T} \]

- **\( K \)**: strike price
- **\( F_0 \)**: forward value of variable
- **\( \sigma \)**: volatility of \( F \)
- **\( T \)**: time when variable is observed
- **\( T^* \)**: time of payoff
Validity of Black’s Model

Two assumptions:
1. The expected value of the underlying variable is its forward price
2. We can discount expected payoffs at rate observed in the market today

It turns out that these assumptions offset each other in the applications of Black’s model that we will consider
Black’s Model for European Bond Options

- Assume that the future bond price is lognormal

\[ c = P(0,T)[F_B N(d_1) - KN(d_2)] \]
\[ p = P(0,T)[KN(-d_2) - F_B N(-d_1)] \]
\[ d_1 = \frac{\ln(F_B / K) + \sigma_B^2 T / 2}{\sigma_B \sqrt{T}} \]
\[ d_2 = d_1 - \sigma_B \sqrt{T} \]

- Both the bond price and the strike price should be cash prices not quoted prices
Approximate duration relation between forward bond price, $F_B$, and forward bond yield, $y_F$

$$\frac{\Delta F_B}{F_B} \approx -D \Delta y_F \quad \text{or} \quad \frac{\Delta F_B}{F_B} \approx -D y_F \frac{\Delta y_F}{y_F}$$

where $D$ is the (modified) duration of the forward bond at option maturity
Yield Vols vs Price Vols (Equation 26.8, page 617)

- This relationship implies the following approximation
  \[ \sigma_B = D_y y_0 \sigma_y \]
  where \(\sigma_y\) is the forward yield volatility, \(\sigma_B\) is the forward price volatility, and \(y_0\) is today’s forward yield.

- Often \(\sigma_y\) is quoted with the understanding that this relationship will be used to calculate \(\sigma_B\).
Theoretical Justification for Bond Option Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time $T$, the option price is

$$P(0,T)E_T[\max(B_T - K, 0)]$$

Also

$$E_T[B_T] = F_0$$

This leads to Black's model
Caps and Floors

- A cap is a portfolio of call options on LIBOR. It has the effect of guaranteeing that the interest rate in each of a number of future periods will not rise above a certain level.
  
  Payoff at time $t_{k+1}$ is $L \delta_k \max(R_k - R_K, 0)$ where $L$ is the principal, $\delta_k = t_{k+1} - t_k$, $R_K$ is the cap rate, and $R_k$ is the rate at time $t_k$ for the period between $t_k$ and $t_{k+1}$.

- A floor is similarly a portfolio of put options on LIBOR. Payoff at time $t_{k+1}$ is $L \delta_k \max(R_K - R_k, 0)$
Caplets

- A cap is a portfolio of “caplets”
- Each caplet is a call option on a future LIBOR rate with the payoff occurring in arrears
- When using Black’s model we assume that the interest rate underlying each caplet is lognormal
Black’s Model for Caps
(Equation 26.13, p. 621)

- The value of a caplet, for period \((t_k, t_{k+1})\) is

\[
L \delta_k P(0, t_{k+1}) [F_k N(d_1) - R_K N(d_2)]
\]

where

\[
d_1 = \frac{\ln(F_k / R_K) + \sigma_k^2 t_k / 2}{\sigma_k \sqrt{t_k}}
\]

and

\[
d_2 = d_1 - \sigma \sqrt{t_k}
\]

- \(F_k\): forward interest rate for \((t_k, t_{k+1})\)
- \(\sigma_k\): forward rate volatility
- \(L\): principal
- \(R_K\): cap rate
- \(\delta_k = t_{k+1} - t_k\)
When Applying Black’s Model To Caps We Must ...

- EITHER
  - Use spot volatilities
  - Volatility different for each caplet
- OR
  - Use flat volatilities
  - Volatility same for each caplet within a particular cap but varies according to life of cap
Theoretical Justification for Cap Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time $t_{k+1}$ the option price is

$$P(0,t_{k+1})E_{k+1}[\max(R_k - R_K,0)]$$

Also

$$E_{k+1}[R_k] = F_k$$

This leads to Black's model
Swaptions

- A swaption or swap option gives the holder the right to enter into an interest rate swap in the future

- Two kinds
  - The right to pay a specified fixed rate and receive LIBOR
  - The right to receive a specified fixed rate and pay LIBOR
Black’s Model for European Swaptions

- When valuing European swap options it is usual to assume that the swap rate is lognormal.

- Consider a swaption which gives the right to pay $s_K$ on an $n$-year swap starting at time $T$. The payoff on each swap payment date is

$$\frac{L}{m} \max(s_T - s_K, 0)$$

where $L$ is principal, $m$ is payment frequency and $s_T$ is market swap rate at time $T$. 


Black’s Model for European Swaptions continued (Equation 26.15, page 627)

The value of the swaption is

\[ LA[s_0 N(d_1) - s_K N(d_2)] \]

where \( d_1 = \frac{\ln(s_0 / s_K) + \sigma^2 T / 2}{\sigma \sqrt{T}} \); \( d_2 = d_1 - \sigma \sqrt{T} \)

\( s_0 \) is the forward swap rate; \( \sigma \) is the swap rate volatility; \( t_i \) is the time from today until the \( i \) th swap payment; and

\[ A = \frac{1}{m} \sum_{i=1}^{m} n P(0, t_i) \]
Theoretical Justification for Swap Option Model

Working in a world that is FRN wrt the annuity underlying the swap, the option price is

\[ \text{LAE}_A[\max(s_T - s_K, 0)] \]

Also

\[ E_A[s_T] = s_0 \]

This leads to Black's model
Relationship Between Swaptions and Bond Options

- An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond.
- A swaption or swap option is therefore an option to exchange a fixed-rate bond for a floating-rate bond.
Relationship Between Swaptions and Bond Options (continued)

- At the start of the swap the floating-rate bond is worth par so that the swaption can be viewed as an option to exchange a fixed-rate bond for par
- An option on a swap where fixed is paid and floating is received is a put option on the bond with a strike price of par
- When floating is paid and fixed is received, it is a call option on the bond with a strike price of par
Deltas of Interest Rate Derivatives

Alternatives:

- Calculate a DV01 (the impact of a 1bps parallel shift in the zero curve)
- Calculate impact of small change in the quote for each instrument used to calculate the zero curve
- Divide zero curve (or forward curve) into buckets and calculate the impact of a shift in each bucket
- Carry out a principal components analysis for changes in the zero curve. Calculate delta with respect to each of the first two or three factors
Convexity, Timing, and Quanto Adjustments

Chapter 27
Forward Yields and Forward Prices

- We define the forward yield on a bond as the yield calculated from the forward bond price.
- There is a non-linear relation between bond yields and bond prices.
- It follows that when the forward bond price equals the expected future bond price, the forward yield does not necessarily equal the expected future yield.
Relationship Between Bond Yields and Prices (Figure 27.1, page 636)
Convexity Adjustment for Bond Yields (Eqn 27.1, p. 637)

- Suppose a derivative provides a payoff at time $T$ dependent on a bond yield, $y_T$ observed at time $T$. Define:
  - $G(y_T)$: price of the bond as a function of its yield
  - $y_0$: forward bond yield at time zero
  - $\sigma_y$: forward yield volatility
- The expected bond price in a world that is FRN wrt $P(0,T)$ is the forward bond price
- The expected bond yield in a world that is FRN wrt $P(0,T)$ is
  \[
  \text{Forward Bond Yield} - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}
  \]
Convexity Adjustment for Swap Rate

The expected value of the swap rate for the period $T$ to $T+\tau$ in a world that is FRN wrt $P(0,T)$ is

\[
\text{Forward Swap Rate} - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}
\]

where $G(y)$ defines the relationship between price and yield for a bond lasting between $T$ and $T+\tau$ that pays a coupon equal to the forward swap rate.
Example 27.1 (page 638)

- An instrument provides a payoff in 3 years equal to the 1-year zero-coupon rate multiplied by $1000
- Volatility is 20%
- Yield curve is flat at 10% (with annual compounding)
- The convexity adjustment is 10.9 bps so that the value of the instrument is \( \frac{101.09}{1.13^3} = 75.95 \)
Example 27.2 (Page 638-639)

- An instrument provides a payoff in 3 years = to the 3-year swap rate multiplied by $100
- Payments are made annually on the swap
- Volatility is 22%
- Yield curve is flat at 12% (with annual compounding)
- The convexity adjustment is 36 bps so that the value of the instrument is $12.36/1.12^3 = 8.80$
Timing Adjustments (Equation 27.4, page 640)

The expected value of a variable, $V$, in a world that is FRN wrt $P(0,T^*)$ is the expected value of the variable in a world that is FRN wrt $P(0,T)$ multiplied by

$$\exp\left[\frac{-\rho VR \sigma V \sigma R R_0 (T^* - T)}{1 + R_0/m} T \right]$$

where $R$ is the forward interest rate between $T$ and $T^*$ expressed with a compounding frequency of $m$, $\sigma_R$ is the volatility of $R$, $R_0$ is the value of $R$ today, $\sigma_V$ is the volatility of $F$, and $\rho$ is the correlation between $R$ and $V$.
Example 27.3 (page 640)

- A derivative provides a payoff 6 years equal to the value of a stock index in 5 years. The interest rate is 8% with annual compounding.
- 1200 is the 5-year forward value of the stock index.
- This is the expected value in a world that is FRN wrt $P(0,5)$.
- To get the value in a world that is FRN wrt $P(0,6)$ we multiply by 1.00535.
- The value of the derivative is $1200 \times 1.00535/(1.08^6)$. 

Quantos
(Section 27.3, page 641)

- Quantos are derivatives where the payoff is defined using variables measured in one currency and paid in another currency.
- Example: contract providing a payoff of $S_T - K$ dollars ($) where $S$ is the Nikkei stock index (a yen number).
Diff Swap

- Diff swaps are a type of quanto
- A floating rate is observed in one currency and applied to a principal in another currency
Quanto Adjustment (page 642)

- The expected value of a variable, $V$, in a world that is FRN wrt $P_X(0,T)$ is its expected value in a world that is FRN wrt $P_Y(0,T)$ multiplied by $\exp(\rho_{VW} \sigma_V \sigma_W T)$ where $W$ is the forward exchange rate (units of $Y$ per unit of $X$) and $\rho_{VW}$ is the correlation between $V$ and $W$. 
Example 27.4 (page 642)

- Current value of Nikkei index is 15,000
- This gives one-year forward as 15,150.75
- Suppose the volatility of the Nikkei is 20%, the volatility of the dollar-yen exchange rate is 12% and the correlation between the two is 0.3
- The one-year forward value of the Nikkei for a contract settled in dollars is $15,150.75e^{0.3 \times 0.2 \times 0.12 \times 1}$ or 15,260.23
Quantos continued

When we move from the traditional risk neutral world in currency $Y$ to the traditional risk neutral world in currency $X$, the growth rate of a variable $V$ increases by

$$\rho \sigma_V \sigma_S$$

where $\sigma_V$ is the volatility of $V$, $\sigma_S$ is the volatility of the exchange rate (units of $Y$ per unit of $X$), and $\rho$ is the coefficient of correlation between the two.
Siegel’s Paradox

An exchange rate $S$ (units of currency $Y$ per unit of currency $X$) follows the risk-neutral process

$$dS = [r_Y - r_X]Sdt + \sigma_S Sdz$$

This implies from Ito's lemma that

$$d(1/S) = [r_X - r_Y + \sigma^2_S](1/S)dt - \sigma_S (1/S)dz$$

Given that the process for $S$ has a drift rate of $r_Y - r_X$, we expect the process for $1/S$ to have a drift of $r_X - r_Y$.

Can you explain this?
When is a Convexity, Timing, or Quanto Adjustment Necessary

- A convexity or timing adjustment is necessary when interest rates are used in a nonstandard way for the purposes of defining a payoff.
- No adjustment is necessary for a vanilla swap, a cap, or a swap option.
Interest Rate Derivatives: Models of the Short Rate

Chapter 28
Black’s model is concerned with describing the probability distribution of a single variable at a single point in time.

A term structure model describes the evolution of the whole yield curve.
The Zero Curve

- The process for the instantaneous short rate, $r$, in the traditional risk-neutral world defines the process for the whole zero curve in this world.
- If $P(t, T)$ is the price at time $t$ of a zero-coupon bond maturing at time $T$,

$$P(t, T) = \hat{E} \left[ e^{-\bar{r}(T-t)} \right]$$

where $\bar{r}$ is the average $r$ between times $t$ and $T$. 
Equilibrium Models

Rendleman & Bartter:
\[ dr = \mu r \, dt + \sigma r \, dz \]

Vasicek:
\[ dr = a(b - r) \, dt + \sigma dz \]

Cox, Ingersoll, & Ross (CIR):
\[ dr = a(b - r) \, dt + \sigma \sqrt{r} \, dz \]
Mean Reversion
(Figure 28.1, page 651)

HIGH interest rate has negative trend

LOW interest rate has positive trend

Reversion Level
Alternative Term Structures in Vasicek & CIR
(Figure 28.2, page 652)
Equilibrium vs No-Arbitrage Models

- In an equilibrium model today’s term structure is an output
- In a no-arbitrage model today’s term structure is an input
Developing No-Arbitrage Model for $r$

A model for $r$ can be made to fit the initial term structure by including a function of time in the drift
Ho and Lee

\[ dr = \theta(t) dt + \sigma dz \]

- Many analytic results for bond prices and option prices
- Interest rates normally distributed
- One volatility parameter, \( \sigma \)
- All forward rates have the same standard deviation
Diagrammatic Representation of Ho and Lee (Figure 28.3, page 655)
Hull and White Model

\[ dr = \left[ \theta(t) - ar \right] dt + \sigma dz \]

- Many analytic results for bond prices and option prices
- Two volatility parameters, \( a \) and \( \sigma \)
- Interest rates normally distributed
- Standard deviation of a forward rate is a declining function of its maturity
Diagrammatic Representation of Hull and White (Figure 28.4, page 656)
Black-Karasinski Model (equation 28.18)

\[ d \ln(r) = \left[ \theta(t) - a(t)r \ln(r) \right] dt + \sigma(t) dz \]

- Future value of \( r \) is lognormal
- Very little analytic tractability
Options on Zero-Coupon Bonds
(equation 28.20, page 658)

- In Vasicek and Hull-White model, price of call maturing at $T$ on a bond lasting to $s$ is
  
  $$LP(0,s)N(h)-KP(0,T)N(h-\sigma_P)$$

- Price of put is
  
  $$KP(0,T)N(-h+\sigma_P)-LP(0,s)N(h)$$

where

$$h = \frac{1}{\sigma_P} \ln \frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2}$$

$$\sigma_P = \frac{\sigma}{a} \left[ 1 - e^{-a(s-T)} \right] \sqrt{\frac{1-e^{-2aT}}{2a}}$$

$L$ is the principal and $K$ is the strike price.

For Ho-Lee $\sigma_P = \sigma(s-T)\sqrt{T}$
Options on Coupon Bearing Bonds

- In a one-factor model a European option on a coupon-bearing bond can be expressed as a portfolio of options on zero-coupon bonds.
- We first calculate the critical interest rate at the option maturity for which the coupon-bearing bond price equals the strike price at maturity.
- The strike price for each zero-coupon bond is set equal to its value when the interest rate equals this critical value.
Interest Rate Trees vs Stock Price Trees

- The variable at each node in an interest rate tree is the \( \Delta t \)-period rate.
- Interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node.
Two-Step Tree Example
(Figure 28.6, page 660))

Payoff after 2 years is \( \text{MAX}[100(r - 0.11), 0] \)
\( p_u = 0.25; \ p_m = 0.5; \ p_d = 0.25; \) Time step = 1yr

\[
\begin{align*}
* & : \ (0.25 \times 3 + 0.50 \times 1 + 0.25 \times 0) e^{-0.12 \times 1} \\
** & : \ (0.25 \times 1.11 + 0.50 \times 0.23 + 0.25 \times 0) e^{-0.10 \times 1}
\end{align*}
\]
Alternative Branching Processes in a Trinomial Tree
(Figure 28.7, page 661)

(a)  (b)  (c)
Procedure for Building Tree

\[ dr = [\theta(t) - ar]dt + \sigma dz \]

1. Assume \( \theta(t) = 0 \) and \( r(0) = 0 \)

2. Draw a trinomial tree for \( r \) to match the mean and standard deviation of the process for \( r \)

3. Determine \( \theta(t) \) one step at a time so that the tree matches the initial term structure
Example (page 662 to 667)

\[ \sigma = 0.01 \]
\[ \alpha = 0.1 \]
\[ \Delta t = 1 \text{ year} \]

The zero curve is as shown in Table 28.1 on page 665
Building the First Tree for the $\Delta t$ rate $R$

- Set vertical spacing $\Delta R = \sigma \sqrt{3\Delta t}$
- Change branching when $j_{\text{max}}$ nodes from middle where $j_{\text{max}}$ is smallest integer greater than $0.184/(a\Delta t)$
- Choose probabilities on branches so that mean change in $R$ is $-aR\Delta t$ and S.D. of change is $\sigma \sqrt{\Delta t}$
The Initial Tree
(Figure 28.8, page 663)

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>R</td>
<td>0.000%</td>
<td>1.732%</td>
<td>0.000%</td>
<td>-1.732%</td>
<td>3.464%</td>
<td>1.732%</td>
<td>0.000%</td>
<td>-1.732%</td>
<td>-3.464%</td>
</tr>
<tr>
<td>p_u</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.8867</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.0867</td>
</tr>
<tr>
<td>p_m</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
</tr>
<tr>
<td>p_d</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.0867</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
Shifting Nodes

- Work forward through tree
- Remember $Q_{ij}$ the value of a derivative providing a $1$ payoff at node $j$ at time $i\Delta t$
- Shift nodes at time $i\Delta t$ by $\alpha_i$ so that the $(i+1)\Delta t$ bond is correctly priced
The Final Tree
(Figure 28.9, Page 665)

---

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>3.824%</td>
<td>6.937%</td>
<td>5.205%</td>
<td>3.473%</td>
<td>9.716%</td>
<td>7.984%</td>
<td>6.252%</td>
<td>4.520%</td>
<td>2.788%</td>
</tr>
<tr>
<td>R</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.8867</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.0867</td>
</tr>
<tr>
<td>p u</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
</tr>
<tr>
<td>p d</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.0867</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
Extensions

The tree building procedure can be extended to cover more general models of the form:

\[ df(r) = [\theta(t) - a f(r)]dt + \sigma dz \]

We set \( x = f(r) \) and proceed similarly to before.
Calibration to determine $a$ and $\sigma$

- The volatility parameters $a$ and $\sigma$ are chosen so that the model fits the prices of actively traded instruments such as caps and European swap options as closely as possible.

- We minimize a function of the form

$$\sum_{i=1}^{n} (U_i - V_i)^2 + P$$

where $U_i$ is the market price of the $i$th calibrating instrument, $V_i$ is the model price of the $i$th calibrating instrument and $P$ is a function that penalizes big changes or curvature in $a$ and $\sigma$. 
HJM Model: Notation

\[ P(t,T) : \text{ price at time } t \text{ of a discount bond with principal of } \$1 \text{ maturing at } T \]

\[ \Omega_t : \text{ vector of past and present values of interest rates and bond prices at time } t \text{ that are relevant for determining bond price volatilities at that time} \]

\[ \nu(t,T,\Omega_t) : \text{ volatility of } P(t,T) \]
Notation continued

\[ f(t,T_1,T_2) \]: forward rate as seen at \( t \) for the period between \( T_1 \) and \( T_2 \)

\[ F(t,T) \]: instantaneous forward rate as seen at \( t \) for a contract maturing at \( T \)

\( r(t) \): short-term risk-free interest rate at \( t \)

\( dz(t) \): Wiener process driving term structure movements
Modeling Bond Prices (Equation 29.1, page 680)

\[ dP(t, T) = r(t)P(t, T)dt + \nu(t, T, \Omega_t)P(t, T)dz(t) \]

We can choose any \( \nu \) function providing

\[ \nu(t, t, \Omega_t) = 0 \quad \text{for all } t \]

Because

\[ f(t, T_1, T_2) = \frac{\ln[P(t, T_1)] - \ln[P(t, T_2)]}{T_2 - T_1} \]

we can use Ito's lemma to determine the process for \( f(t, T_1, T_2) \). Letting \( T_2 \) approach \( T_1 \) we get a process for \( F(t, T) \).
The process for \( F(t,T) \)
Equation 29.4 and 29.5, page 681)

\[
dF(t,T) = \nu(t,T,\Omega_t)\nu_T(t,T,\Omega_t)dt - \nu_T(t,T,\Omega_t)dz(t)
\]

This result means that if we write
\[
dF(t,T) = m(t,T,\Omega_t)dt + s(t,T,\Omega_t)dz
\]
we must have
\[
m(t,T,\Omega_t) = s(t,T,\Omega_t)\int_t^T s(t,\tau,\Omega_t) d\tau
\]
Similar results hold when there is more than one factor
Tree Evolution of Term Structure is Non-Recombining

Tree for the short rate \( r \) is non-Markov
(see Figure 29.1, page 682)
The LIBOR market model is a model constructed in terms of the forward rates underlying caplet prices.
Notation

\( t_k \): \( k \)th reset date

\( F_k(t) \): forward rate between times \( t_k \) and \( t_{k+1} \)

\( m(t) \): index for next reset date at time \( t \)

\( \varsigma_k(t) \): volatility of \( F_k(t) \) at time \( t \)

\( \nu_k(t) \): volatility of \( P(t, t_k) \) at time \( t \)

\( \delta_k \): \( t_{k+1} - t_k \)
Volatility Structure

We assume a stationary volatility structure where the volatility of $F_k(t)$ depends only on the number of accrual periods between the next reset date and $t_k$ [i.e., it is a function only of $k - m(t)$]
In Theory the \( \Lambda \)'s can be determined from Cap Prices

Define \( \Lambda_i \) as the volatility of \( F_k(t) \) when \( k - m(t) = i \)

If \( \sigma_k \) is the volatility for the \( (t_k, t_{k+1}) \) caplet. If the model provides a perfect fit to cap prices we must have

\[
\sigma_k^2 t_k = \sum_{i=1}^{k} \Lambda_{k-i}^2 \delta_{i-1}
\]

This allows the \( \Lambda \)'s to be determined inductively
Example 29.1 (Page 684)

- If Black volatilities for the first three caplets are 24%, 22%, and 20%, then
  \[ \Lambda_0 = 24.00\% \]
  \[ \Lambda_1 = 19.80\% \]
  \[ \Lambda_2 = 15.23\% \]
Example 29.2 (Page 684)

<table>
<thead>
<tr>
<th>n</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</tr>
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<tr>
<td>$\sigma_n(%)$</td>
<td>15.50</td>
<td>18.25</td>
<td>17.91</td>
<td>17.74</td>
<td>17.27</td>
</tr>
<tr>
<td>$\Lambda_{n-1}(%)$</td>
<td>15.50</td>
<td>20.64</td>
<td>17.21</td>
<td>17.22</td>
<td>15.25</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>n</th>
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<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma_n(%)$</td>
<td>16.79</td>
<td>16.30</td>
<td>16.01</td>
<td>15.76</td>
<td>15.54</td>
</tr>
<tr>
<td>$\Lambda_{n-1}(%)$</td>
<td>14.15</td>
<td>12.98</td>
<td>13.81</td>
<td>13.60</td>
<td>13.40</td>
</tr>
</tbody>
</table>
The Process for $F_k$ in a One-Factor LIBOR Market Model

\[ dF_k = \ldots + \Lambda_{k-m(t)} F_k \, dz \]

The drift depends on the world chosen. In a world that is forward risk-neutral with respect to $P(t, t_{i+1})$, the drift is zero.
It is often convenient to choose a world that is always FRN wrt a bond maturing at the next reset date. In this case, we can discount from \( t_{i+1} \) to \( t_i \) at the \( \delta_i \) rate observed at time \( t_i \). The process for \( F_k \) is

\[
\frac{dF_k}{F_k} = \sum_{j=m(t)}^{i} \frac{\delta_i F_i \Lambda_{i-m(t)} \Lambda_{k-m(t)}}{1 + \delta_i F_i} dt + \Lambda_{k-m(t)} dz
\]
The LIBOR Market Model and HJM

In the limit as the time between resets tends to zero, the LIBOR market model with rolling forward risk neutrality becomes the HJM model in the traditional risk-neutral world.
We assume no change to the drift between reset dates so that

\[
F_k(t_{j+1}) = F_k(t_j) \exp \left[ \left( \sum_{j=k}^{i} \frac{\delta_i F(t_j) \Lambda_{i-j} \Lambda_{k-j}}{1 + \delta_j L_j} - \frac{\Lambda_{k-j}^2}{2} \right) \delta_k + \Lambda_{k-j} \varepsilon \sqrt{\delta_j} \right]
\]
Multifactor Versions of LMM

- LMM can be extended so that there are several components to the volatility.
- A factor analysis can be used to determine how the volatility of $F_k$ is split into components.
Ratchet Caps, Sticky Caps, and Flexi Caps

- A plain vanilla cap depends only on one forward rate. Its price is not dependent on the number of factors.
- Ratchet caps, sticky caps, and flexi caps depend on the joint distribution of two or more forward rates. Their prices tend to increase with the number of factors.
There is an analytic approximation that can be used to value European swap options in the LIBOR market model. See equations 29.18 and 29.19 on page 689.
Calibrating the LIBOR Market Model

- In theory the LMM can be exactly calibrated to cap prices as described earlier.
- In practice we proceed as for short rate models to minimize a function of the form

\[ \sum_{i=1}^{n} (U_i - V_i)^2 + P \]

where \( U_i \) is the market price of the \( i \)th calibrating instrument, \( V_i \) is the model price of the \( i \)th calibrating instrument and \( P \) is a function that penalizes big changes or curvature in \( a \) and \( \sigma \).
Types of Mortgage-Backed Securities (MBSs)

- Pass-Through
- Collateralized Mortgage Obligation (CMO)
- Interest Only (IO)
- Principal Only (PO)
Option-Adjusted Spread (OAS)

- To calculate the OAS for an interest rate derivative, we value it assuming that the initial yield curve is the Treasury curve plus a spread.
- We use an iterative procedure to calculate the spread that makes the derivative’s model price equal to the market price. This is the OAS.
Chapter 30

Swaps Revisited
Valuation of Swaps

- The standard approach is to assume that forward rates will be realized.
- This works for plain vanilla interest rate and plain vanilla currency swaps, but does not necessarily work for non-standard swaps.
Variations on Vanilla Interest Rate Swaps

- Principal different on two sides
- Payment frequency different on two sides
- Can be floating-for-floating instead of floating-for-fixed
- It is still correct to assume that forward rates are realized
- How should a swap exchanging the 3-month LIBOR for 3-month CP rate be valued?
Compounding Swaps (Business Snapshot 30.2, page 699)

- Interest is compounded instead of being paid
- Example: the fixed side is 6% compounded forward at 6.3% while the floating side is LIBOR plus 20 bps compounded forward at LIBOR plus 10 bps.
- This type of compounding swap can be valued using the “assume forward rates are realized” rule. This is because we can enter into a series of forward contracts that have the effect of exchanging cash flows for their values when forward rates are realized.
Currency Swaps

- Standard currency swaps can be valued using the “assume forward LIBOR rate are realized” rule.
- Sometimes banks make a small adjustment because LIBOR in currency A is exchanged for LIBOR plus a spread in currency B.
More Complex Swaps

- LIBOR-in-arrears swaps
- CMS and CMT swaps
- Differential swaps

These cannot be accurately valued by assuming that forward rates will be realized.
LIBOR-in Arrears Swap (Equation 30.1, page 701)

- Rate is observed at time $t_i$ and paid at time $t_i$ rather than time $t_{i+1}$
- It is necessary to make a convexity adjustment to each forward rate underlying the swap
- Suppose that $F_i$ is the forward rate between time $t_i$ and $t_{i+1}$ and $\sigma_i$ is its volatility
- We should increase $F_i$ by

$$F_i \left( 1 + F_i \tau_i \right) \left( t_{i+1} - t_i \right)$$

when valuing a LIBOR-in-arrears swap
CMS swaps

- Swap rate observed at time $t_i$ is paid at time $t_{i+1}$
- We must
  - make a convexity adjustment because payments are swap rates (= yield on a par yield bond)
  - Make a timing adjustment because payments are made at time $t_{i+1}$ not $t_i$
- See equation 30.2 on page 702
Differential Swaps

- Rate is observed in currency Y and applied to a principal in currency X
- We must make a quanto adjustment to the rate
- See equation 30.3 on page 704.
Equity Swaps (page 704-705)

- Total return on an equity index is exchanged periodically for a fixed or floating return.
- When the return on an equity index is exchanged for LIBOR the value of the swap is always zero immediately after a payment. This can be used to value the swap at other times.
Swaps with Embedded Options
(page 705-708)

- Accrual swaps
- Cancelable swaps
- Cancelable compounding swaps
Other Swaps (page 708-709)

- Indexed principal swap
- Commodity swap
- Volatility swap
- Bizarre deals (for example, the P&G 5/30 swap in Business Snapshot 30.4 on page 709)
Real Options
Chapter 31
An Alternative to the NPV Rule for Capital Investments

- Define stochastic processes for the key underlying variables and use risk-neutral valuation
- This approach (known as the real options approach) is likely to do a better job at valuing growth options, abandonment options, etc than NPV
The Problem with using NPV to Value Options

- Consider the example from Chapter 11: risk-free rate = 12%; strike price = $21

Stock Price = $22

$20

Stock price = $20

Stock Price=$18

- Suppose that the expected return required by investors in the real world on the stock is 16%. What discount rate should we use to value an option with strike price $21?
Correct Discount Rates are Counter-Intuitive

- Correct discount rate for a call option is 42.6%
- Correct discount rate for a put option is −52.5%
General Approach to Valuation

- We can value any asset dependent on a variable $\theta$ by
  - Reducing the expected growth rate of $\theta$ by $\lambda s$
    where $\lambda$ is the market price of $\theta$-risk and $s$ is the volatility of $\theta$
  - Assuming that all investors are risk-neutral
Extension to Many Underlying Variables

- When there are several underlying variables $\theta_i$, we reduce the growth rate of each one by its market price of risk times its volatility and then behave as though the world is risk-neutral.
- Note that the variables do not have to be prices of traded securities.

The market price of risk of a variable is given by

$$\lambda = \frac{\rho}{\sigma_m} (\mu_m - r)$$

where $\rho$ is the instantaneous correlation between percentage changes in the variable and returns on the market; $\sigma_m$ is the volatility of the market's return; $\mu_m$ is the expected return on the market; and $r$ is the short-term risk-free rate.
Example of Application of Real Options Approach to Valuing Amazon.com (Business Snapshot 31.1; Schwartz and Moon)

- Estimate stochastic processes for the company’s sales revenue and its average growth rate.
- Estimated the market price of risk and other key parameters (cost of goods sold as a percent of sales, variable expenses as a percent of sales, fixed expenses, etc.)
- Use Monte Carlo simulation to generate different scenarios in a risk-neutral world.
- The stock price is the average of the present values of the net cash flows discounted at the risk-free rate.
Commodity Prices

- Futures prices can be used to define the process followed by a commodity price in a risk-neutral world.
- We can build in mean reversion and use a process for constructing trinomial trees that is analogous to that used for interest rates in Chapter 28.
Example (page 671)

A company has to decide whether to invest $15 million to obtain 6 million barrels of oil at the rate of 2 million barrels per year for three years. The fixed operating costs are $6 million per year and the variable costs are $17 per barrel. The spot price of oil $20 per barrel and 1, 2, and 3-year futures prices are $22, $23, and $24, respectively. The risk-free rate is 10% per annum for all maturities.
The Process for Oil

We assume that this is

\[ d \ln(S) = \left[ \theta(t) - \alpha \ln(S) \right] dt + \sigma \, dz \]

Where \( \alpha = 0.1 \) and \( \sigma = 0.2 \)
Tree Assuming $\theta(t) = 0$; Fig 31.1
Final Tree for Oil Prices; Fig 31.2
Valuation of Base Project; Fig 31.3

![Diagram of valuation process with nodes and values]

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
<th>H</th>
<th>I</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_u )</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.8867</td>
<td>0.1217</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.0867</td>
</tr>
<tr>
<td>( p_m )</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
<td>0.6566</td>
<td>0.6666</td>
<td>0.6566</td>
<td>0.0266</td>
</tr>
<tr>
<td>( p_d )</td>
<td>0.1667</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.0867</td>
<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
Valuation of Option to Abandon; Fig 31.4
(No Salvage Value; No Further Payments)

Options, Futures, and Other Derivatives, 6th Edition, Copyright © John C. Hull 2005
Value of Expansion Option; Fig 31.5 (Company Can Increase Scale of Project by 20% for $2 million)

<table>
<thead>
<tr>
<th>Node</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
<th>G</th>
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<td>0.2217</td>
<td>0.1667</td>
<td>0.1217</td>
<td>0.8867</td>
</tr>
</tbody>
</table>
Derivatives Mishaps and What We Can Learn from Them

Chapter 32
Big Losses by Financial Institutions

- Allied Irish Bank ($700 million)
- Barings ($1 billion)
- Daiwa ($1 billion)
- Kidder Peabody ($350 million)
- LTCM ($4 billion)
- Midland Bank ($500 million)
- National Westminster Bank ($130 million)
Big Losses by Non-Financial Corporations

- Allied Lyons ($150 million)
- Gibsons Greetings ($20 million)
- Hammersmith and Fulham ($600 million)
- Metallgesellschaft ($1.8 billion)
- Orange County ($2 billion)
- Procter and Gamble ($90 million)
- Shell ($1 billion)
- Sumitomo ($2 billion)
Lessons for All Users of Derivatives

- Risk must be quantified and risk limits set
- Exceeding risk limits not acceptable even when profits result
- Do not assume that a trader with a good track record will always be right
- Be diversified
- Scenario analysis and stress testing is important
Lessons for Financial Institutions

- Do not give too much independence to star traders
- Separate the front middle and back office
- Models can be wrong
- Be conservative in recognizing inception profits
- Do not sell clients inappropriate products
- Liquidity risk is important
- There are dangers when many are following the same strategy
Lessons for Non-Financial Corporations

- It is important to fully understand the products you trade
- Beware of hedgers becoming speculators
- It can be dangerous to make the Treasurer’s department a profit center