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Income distribution dependence of poverty measure: A theoretical analysis

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Abstract

Using a modified deprivation (or poverty) function, in this paper, we theoretically study the changes in poverty with respect to the 'global' mean and variance of the income distribution using Indian survey data. We show that when the income obeys a log-normal distribution, a rising mean income generally indicates a reduction in poverty while an increase in the variance of the income distribution increases poverty. This altruistic view for a developing economy, however, is not tenable anymore once the poverty index is found to follow a pareto distribution. Here although a rising mean income indicates a reduction in poverty, due to the presence of an inflexion point in the poverty function, there is a critical value of the variance below which poverty decreases with increasing variance while beyond this value, poverty undergoes a steep increase followed by a decrease with respect to higher variance. Identifying this inflexion point as the poverty line, we show that the pareto poverty function satisfies all three standard axioms of a poverty index [N.C. Kakwani, Econometrica 43 (1980) 437; A.K. Sen, Econometrica 44 (1976) 219] whereas the log-normal distribution falls short of this requisite. Following these results, we make quantitative predictions to correlate a developing with a developed economy.

Keywords: Poverty; Inequality; Income distribution; Consumption deprivation; Inflexion point

1. Introduction

Pioneered by the paradigmatic contributions of Sen [1–3] and Atkinson [4], a remarkable amount of effort has been undertaken [5–8] in understanding the economics of poverty and inequality from a theoretical perspective. The studies range from being aptly mathematical in nature to a qualitative characterization of such population dialectics. Pradhan and Ravallion [9] have used qualitative assessments of perceived consumption adequacy based on a household survey. They claim that perceived consumption needs can be a more promising approach than the subjective income-based poverty line. This consumption norm can correspond to a saturation level of consumption, below which the individual could be considered to be in poverty. Further, in this paper, our approach is rather complementary to a lemma-based mathematical model

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in that we use survey-based consumption data to quantify the dependence of a well-known poverty function [7,18] on the mean and variance of the income distribution. To this end, we use income-expenditure data from a 'developing nation' (India in our case) and utilize the well established technique of data fitting to define the per capita consumption as a function of income. Here the implicit assumption is that of a near equilibrium situation such that the time dependence of both income and consumption variables can be considered as transients without much effect on the asymptotic distributions. Deaton [10] has discussed the ambiguity that arises using survey data versus national accounts data for individual consumption or income levels. Although the survey consumption data seem to understate the true consumption levels, we are however using the data as a backup of our analytical results thereby restricting our claims to being qualitative in nature. Such comparisons with real data help us have approximate ideas of the values of the unknown parameters, two in our model, although the general conclusions are remarkably independent of these parameter values.

The intrinsic idea of consumption deprivation (CD) helps to quantify an otherwise qualitative feature of any economy, that of poverty. This can be defined through an index, a mathematical measure, that satisfies a set of axioms [2] as discussed later in the article (monotonicity and transfer axioms). Given different shapes of income distribution data curves, a distribution sensitive measure could be appealing because it can more appropriately reflect the extent of deprivation among the poor that may be ignored by other measures of poverty commonly used in the poverty literature, such as head-count ratio and poverty-gap ratio [11]. Using the standard two-parameter-dependent definition of an income distribution, the parameters being the mean and variance of the distribution, respectively, we would consider two distributions—the log-normal distribution and the pareto distribution—to study the effects of changes in the mean and variance of the underlying income distribution on poverty. We find that for the log-normal distribution, an increase in mean income and a reduction in the variance of income distribution can reduce poverty. It also hints toward a trade-off, in that while an increase in average income reduces poverty, a simultaneous increase in income variance can escalate poverty. This result is likely to suggest that reducing income inequality should be the precondition for lowering poverty. These general results are then contrasted in the following section using a different model for the income distribution, the pareto distribution. The objective is basically to probe whether the results obtained are universal in nature and if not, then which distribution defines a better measure of poverty. Such conclusions are a remarkable modification to the general consensus which claims that for low incomes, the distribution is generally expected to be log-normal [12] or exponential [13].[1]

In a later section, we proceed to show that the fundamental reason for which the poverty index defined by the pareto distribution function out-competes the log-normal index is directly related to it being standardized suitably with respect to all three axioms—the *monotonicity axiom*, the *transfer axiom* and the *transfer sensitivity axiom*—of a poverty index measure [2,6,8]. Initially propounded by Sen [2] and Kakwani [6], and later discussed lucidly in an article by Kumar et al. [8], these three axioms are supposed to be the necessary conditions for an ideal poverty index. Our calculations clearly show that although the *transfer axiom* is satisfied by the log-normal distribution, only the pareto distribution satisfies all three—*the monotonicity* and *the transfer sensitivity* axioms. Additionally, we show that the pareto distribution has an inherent line of poverty that can be identified with the inflexion point whereas the log-normal one fails to come up with any such self-sufficient measure. As has been shown by Atkinson [4] that a poverty line need not have a scientific basis and *can be chosen administratively using certain objective criteria* [8], admittedly it is no pre-requisite for a poverty index. However, a poverty line that is inbuilt in the distribution function itself has the clear advantage that instead of having to resort to arbitrary external parameters, one has a self-consistent definition of the reference line of poverty measure directly from the distribution function itself thereby making it more self-sufficient. Indeed many recent theoretical studies are directed toward a quantification of this poverty line [14] and they find a clear signature of exponential behavior above this poverty line much in line with

[1] Sen [2] introduced the notion of deprivation in the income distribution literature, and criticized the use of the head-count ratio as a measure of poverty. Rao (1981) suggested broadening the scope of poverty measurement to nutritional norms as opposed to monetary measures. If poverty is to be regarded as negative welfare, it makes sense to relate it to CD resulting from an uneven income distribution rather than to the income distribution alone as is done by the traditional poverty ratio index [8].

our findings. Once again, such an indirect but quantitative check speaks in favor of the pareto distribution function.

2. Poverty impact of changes in log-normal income distribution

Poverty, by our definition, equals CD of an essential food. The necessity of defining poverty as a multidimensional concept rather than relying on income or consumption expenditures per capita has been well documented. Although it is important to assess deprivation with more than one attribute (see Refs. [15–18]), we consider the case of most essential food item that is required for survival, in an attempt to include deprivation into the poverty index. Such an index would suggest that a person can be considered poor if the individual's consumption falls within the deprivation area in the diagram (see lower panel of Fig. 1), that is, the cumulative difference between the saturation consumption level of cereal and actual cereal consumption by the community as a whole. Here we have reformulated Kumar et al.'s non-linear consumption function [8] to show positive consumption even at zero income level (see upper panel of Fig. 1). However, the non-linear function used in Ref. [8] allows for a saturation level of consumption norm for food-grains. The non-linear function used in our paper with an equivalent saturation level of consumption norm for food-grains is as follows:

$$C(y) = \frac{V \exp(y)}{K + \exp(y)}, \tag{1}$$

where C is the consumption expenditure on food-grains, y is income and the parameters V, $K(>0)$ represent the saturation level of real food-grain consumption expenditure or the bliss level and the level of income needed to consume one half of the saturation level, respectively. The idea can be correlated with a Fermi-type distribution that maps to the economics perspective through the well-known Gini index [19]. CD or poverty (P) can be defined as the shortfall of actual consumption expenditure relative to saturation level V, or $CD = V - C$. Thus, the non-linear CD function is derived as

$$CD = \frac{VK}{K + \exp(y)}. \tag{2}$$

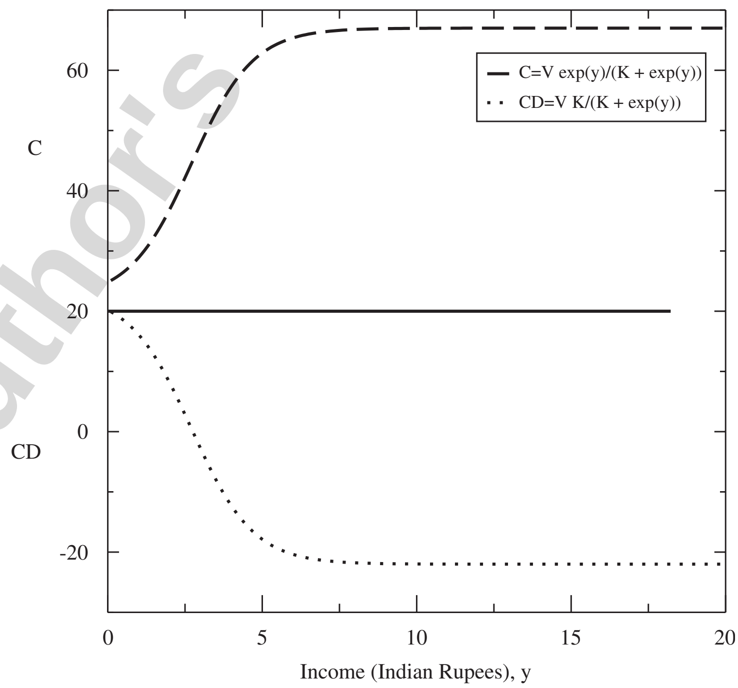

Fig. 1. Consumption $C(y)$ and deprivation $CD(y)$ functions against income y.

This function, being a convex decreasing function of income provides a direct measure of poverty based on nutritional norms, while V and K are parameters of a concave Engel curve. Here $C \to V$ represents the idealistic limit where there is no deprivation or poverty corresponding to a static equilibrium in the social dialectics mathematically represented by $y = y^*$. In what follows, we would consider two asymptotic regimes—$y \to 0$ and $y \to \infty$—physically which correspond to the low and high income groups, respectively. Naturally our focus would be on the $y \to 0$ limit, that is on the low income section although the analysis would encompass both limits.

If consumption of the most essential food item follows a concave non-linear functional form and if individual poverty is measured as the difference between the saturation level of consumption of the essential food item and its actual level, assumption of a log-normal income distribution implies a reduction in poverty with the increase of mean income of the population and an increase in inequality with increasing poverty. This new measure of poverty is based on the notion of CD of a very essential staple food such as rice or wheat (cereal), derived from a non-linear, monotonically increasing concave consumption function varying with the income, albeit with no specific reference to a subjective poverty line. The standard log-normal probability density function (pdf) is defined as

$$f(y/\mu, \sigma^2) = \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right], \tag{3}$$

where y is normally distributed with mean μ and variance σ^2 (both positive real numbers). With this log-normal pdf for the income y, the poverty equation can be rewritten as follows:

$$\begin{aligned} P &= \int_0^\infty CD(y) f(y)\,\mathrm{d}y \\ &= \int_0^\infty \frac{VK}{K + \exp(y)} \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln\, y - \mu)^2}{2\sigma^2}\right] \mathrm{d}y. \end{aligned} \tag{4}$$

Partial derivatives of the above equation (4) with respect to μ and σ^2 give

$$\begin{aligned} \frac{\partial P}{\partial \mu} &= \int_0^\infty \frac{VK}{K + \exp(y)} \frac{1}{y\sigma\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right] \frac{\ln y - \mu}{\sigma^2}\,\mathrm{d}y, \\ \frac{\partial P}{\partial \sigma^2} &= \int_0^\infty \frac{VK}{K + \exp(y)} \frac{1}{2y\sigma^3\sqrt{2\pi}} \exp\left[-\frac{(\ln y - \mu)^2}{2\sigma^2}\right] \left[\frac{(\ln y - \mu)^2}{\sigma^2} - 1\right] \mathrm{d}y. \end{aligned} \tag{5}$$

2.1. Asymptotic solutions of the poverty function

This section deals with asymptotic solutions of the poverty functions for extremely low ($y \to 0$) to moderate values of the income distribution. This is mathematically categorized in the following manner:

For moderate incomes, one can define the consumption function as $C(y) = V\exp(y)/(K + \exp(y)) = C_{\mathrm{mod}}(y)$, say,

1. $C_{\mathrm{mod}}(y \to 0) = \frac{V}{K+1}$ and
2. $C_{\mathrm{mod}}(y \to \infty) = V$,

whereas for very low income groups, $C(y) = Vy/(K + y) = C_{\mathrm{low}}(y)$, say,

1. $C_{\mathrm{low}}(y \to 0) = 0$ and
2. $C_{\mathrm{low}}(y \to \infty) = V$.

The above comparison clearly shows that although both definitions of the consumption function are generally equivalent in the low income limit, for the absolutely needy groups, $C_{\mathrm{mod}}(y)$ predicts a

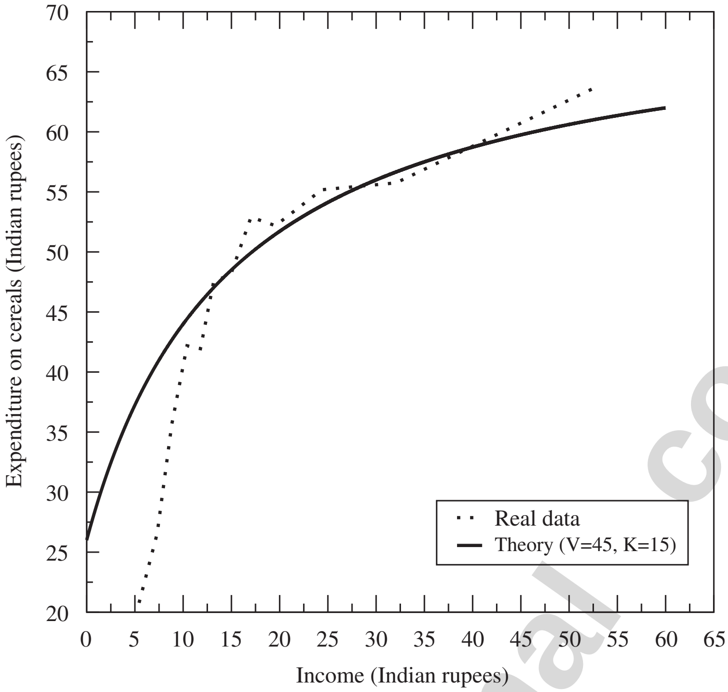

Fig. 2. Consumption $C(y)$ plotted against income y: data fitting to evaluate V and K using data from Indian National Sample survey 1999–2000, 55th round.

non-zero ($V/(K+1)$) lower limit of income which is more realistic than $C_{\text{low}}(y \to 0) = 0$. A linear stability analysis of $C_{\text{low}}(y)$ also shows that $y = 0$ is an unstable fixed point, which further strengthens this conviction. Henceforth, our attention will mainly be focused toward the lowest income groups defined by $C_{\text{low}}(y)$, although we would flip back and forth between the moderate to the low income classes for comparisons. Before proceeding any further, though, we first derive the values of the parameters V, K by fitting the function C_{mod} with actual survey data obtained from National Sample Survey, 1999–2000, 55th Round, India. We would be using these values of V, K in all analysis in this paper. Fig. 2 portrays the shape of an Engle curve, graphing real cereal expenditure against the total expenditure—a surrogate for income.

The above exact data fitting conclusively shows that the parameters V and K have the respective values 45 and 15 in Indian currency (Rupees). These are roughly equivalent to 1.0 and 0.33 USD, respectively. Now using these values, we study the case for typically the lowest income classes defined by the consumption function $C_{\text{low}}(y)$. In this case, however, we need to focus on both low and high limits of the variance. Up to first order in σ^2, we find that

$$P_{\text{low}\,\sigma\to 0} = \frac{V}{K}\left[K - \exp\left(\mu - \frac{\sigma^2}{2}\right)\right]. \tag{6}$$

The poverty dependence on the mean for this asymptotic regime can be understood from Fig. 3.

Fig. 3 tells us that poverty is a monotonically increasing function of variance for a fixed mean (taken to be 2.773 for a direct comparison with Fig. 4 later). On the other hand, for a fixed variance (0.001), poverty decreases with mean. This result is very remarkable but needs to be taken with a pinch of salt, especially since this is true only in the asymptotic ($\sigma \to 0$) regime. We will revisit this issue in the following section where we discuss the situation when both the mean and the variance of the income distribution are simultaneously varying.

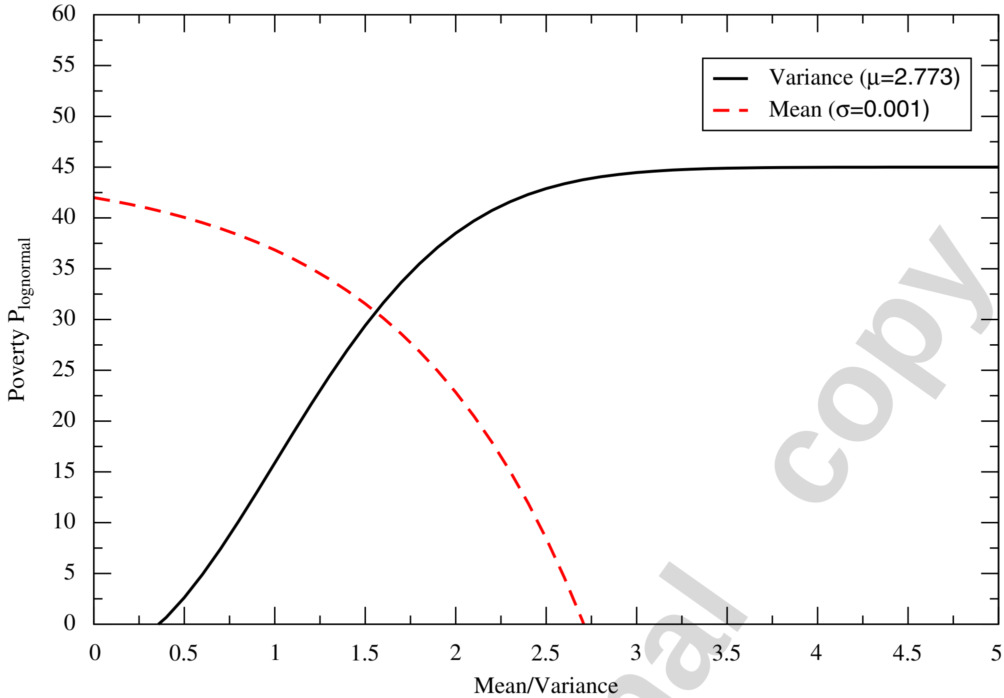

Fig. 3. Poverty versus mean μ (for fixed variance $\sigma^2 = 0.001$) and versus variance σ^2 (for a fixed mean $\mu = 2.773$) for a log-normal distribution for the limiting case $\sigma^2 \to 0$ defined by Eq. (6).

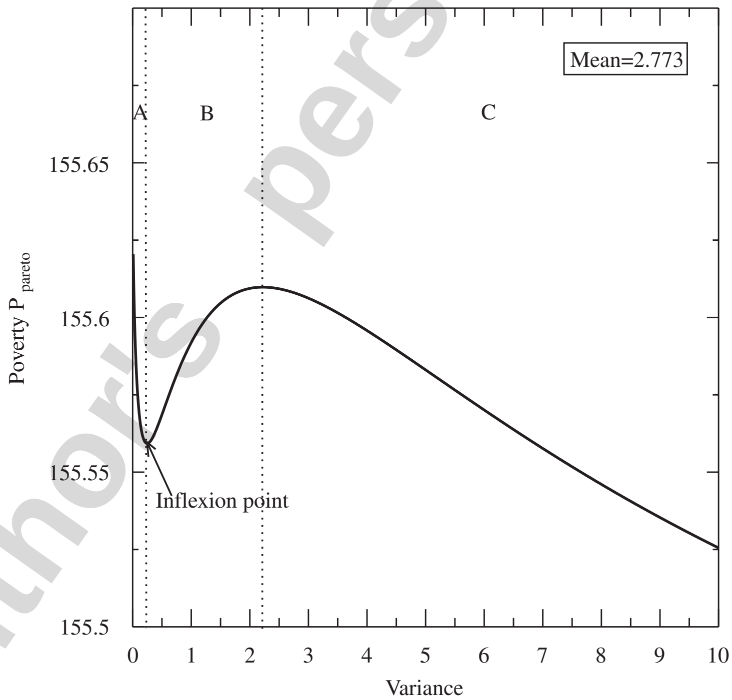

Fig. 4. Poverty versus mean μ for fixed variance in a pareto distribution close to the inflexion point ($\mu = 2.773$). Zone A represents an under-developed economy; zone B defines the poverty—variance relation for a developing nation; zone C represents an economically developed nation.

2.2. Overall impact of simultaneous changes in mean and variance

Here we show what effect any change, either increase or decrease, in the income distribution has on the overall poverty function when the distribution is log-normal and when both mean and variance are varying.

Since our focus is on the low income group, we will be using $C_{\text{low}}(y)$ as our definition for the consumption function. The attention here would be to decipher the joint variation of the poverty function $P(\mu, \sigma^2)$ with respect to μ and σ^2. Once again using a $1/y$ expansion[2] up to the first order, we find that the joint poverty function reads as

$$\begin{aligned} \mathrm{d}P(\mu,\sigma^2) &= \frac{\partial P}{\partial \mu}\,\mathrm{d}\mu + \frac{\partial P}{\partial \sigma^2}\,\mathrm{d}\sigma^2 \\ &= \frac{VK}{\sigma^2}\exp\left[-\left(\mu-\frac{\sigma^2}{2}\right)\right]\mathrm{d}\mu + \frac{VK}{2}\exp\left[-\left(\mu-\frac{\sigma^2}{2}\right)A\right]\mathrm{d}\sigma^2. \end{aligned} \tag{7}$$

This equation suggests that poverty is a decreasing function of changes in μ and an increasing function of changes in σ^2. For a fixed variance, $\mathrm{d}\sigma^2 = 0$, and hence the first component of Ref. [7], reflecting change in μ, will provide convergence; and with a fixed mean, $\mathrm{d}\mu = 0$, the second component, exhibiting change in σ^2, will give convergence of the equation. When both the mean and variance of the income distribution change as a result of changes in macroeconomic policies, their effect on poverty can be evaluated via Eq. (7). The notable point here is the fundamental qualitative difference with the prediction from Eq. (6). As opposed to the earlier asymptotic result where increase of the mean income was expected to generate a positive augmentation in poverty (for fixed variance) followed by a saturation at a particular value μ_c, Eq. (7) with a fixed σ clearly suggests that poverty decreases with increase of the mean income. This apparent dichotomy can be understood once we analyze the physical meaning hidden in Eq. (6). It says that in a relatively large group of low earning population, a very small variance between the earners contributes to an increase in poverty for very low to moderate values of the mean income. However, once the mean income reaches a critical value, this spurious effect saturates off. This can be contrasted with the prediction from the last equation which holds true for moderate to large values of σ. We would like to specifically point out here that both predictions from Eqs. (6) and (7) are true but in their respective regimes defined by small to large values of σ.

3. Poverty impact of changes in pareto income distribution

In this section, our objective is to study the mean and variance dependence of the poverty function, replacing the log-normal probability distribution, previously assumed, with a pareto distribution and contrast the findings later. Once again we would conform to the same definitions of consumption and deprivation functions (1) and (2) and try to understand the qualitative changes in the poverty function of a growing economy with respect to changes in the mean and variance of the overall income distribution.

The standard pareto pdf f_{pareto} defined over the interval $y \geqslant b$ is given by

$$f_{\text{pareto}}(y) = \frac{ab^a}{y^{a+1}}, \tag{8}$$

where the mean μ and the variance σ^2 can be easily shown to be as follows:

$$\begin{aligned} \mu &= \frac{ab}{a-1}, \\ \sigma^2 &= \frac{ab^2}{(a-1)^2(a-2)}. \end{aligned} \tag{9}$$

The pareto exponent a have been found to vary from circa 2.3–2.5 for a first-world economy [20] to 0.81–0.92 for a developing economy like India [21,22]. With the pareto pdf defined above, the poverty function P_{pareto}

[2]This might sound confusing since we are discussing small income but in effect, all that we are doing is to use a well known $1/y$ expansion prevalent in statistical mechanics. It is generally valid for a considerable range involving large to moderate values of the variable y. We have checked this result using C_{mod} and the qualitative results remain altogether unaltered.

reads as follows:

$$\begin{aligned} P_{\text{pareto}}(a,b) &= \int_b^{\infty} CD(y) f_{\text{pareto}}(y)\,\mathrm{d}y \\ &= VKab^a \int_b^{\infty} \frac{\mathrm{d}y}{(K+y)y^{a+1}} \\ &= \frac{V}{K}\left[1 - ab^a \int_b^{\infty} \mathrm{d}y \frac{1}{(K+y)y^a}\right]. \end{aligned} \tag{10}$$

Defining the identity $I(a) = \int_b^{\infty} \mathrm{d}y/(K+y)y^{a+1}$, and taking recourse to a bit of algebra one can deduce a recursive relation

$$I(a) = \frac{1}{Kab^a}\left[1 - \frac{ab}{K(a-1)} + \frac{ab^2}{K^2(a-2)}\right] - \frac{1}{K^3} I(a-3,b). \tag{11}$$

Eq. (11) can be correlated with a hyper-geometric ${}_2F_1$ series[3] and for specified values of the parameters can be solved numerically. For our purpose though, we consider the limit $a \to \infty$ to have a first hand impression of the situation

$$P(a \to \infty, b) = \frac{V}{K}\left[1 - \frac{1}{K}\frac{1}{1+b/K}\right]. \tag{12}$$

We would now directly evaluate the poverty function in a more physical limit. Without any loss of generality we choose the limit $K \to 0$ which is akin to the $1/y$ expansion we did in deriving the poverty function for the log-normal distribution. We would shortly see that in this case, this basic expansion allows us to have an 'exact' derivation of the poverty function as opposed to its log-normal counterpart. Up to the first order in $1/y$ and utilizing Eq. (9), we find

$$P(a,b) = VK\left[\frac{1}{\mu}\frac{a^2}{a^2-1} - \frac{1}{\mu^2}\frac{a^3}{(a-1)^2(a+2)}\right], \tag{13}$$

where $a = 2 + \sqrt{1 + \mu^2/\sigma^2}$ and $b = ((a-1)/a)\mu$. A numerical solution of the above equation (13)[4] shows that it has a pair of inflexion points,[5] out of which the physical pair is at $\mu = 3.05139$ and $\sigma^2 = 0.0692138$. Solving around this inflexion point, we now come across one of the most remarkable results of this article, the fact that poverty initially decreases with increasing variance until it reaches a critical value $\sigma^2 = \sigma_c^2$ beyond which the poverty starts increasing with variance followed by a dip once again.

Fig. 4 has been drawn using $\sigma^2 = 2.773$, a value reasonably close to the inflexion point. The plot shows that poverty decreases until it reaches the point $\mu_c \sim 0.25$ after which it starts increasing approximately until $\sigma^2 = 2$ and then it starts decreasing again. This result is in marked contrast with the log-normal case where the poverty rather uninterestingly decreases with increasing mean for a fixed variance, and increases with variance for a fixed mean. It is now not difficult to pinpoint the detailed meaning of this result. Referring to Fig. 4, zone A defines a rather 'underdeveloped' economy, zone B stands for a 'developing' economy, our case in study, while the final zone C clearly indicates what one would expect in the case of an economically 'developed' nation. We can probably claim without much ambiguities that a pareto distribution has the power to encapsulate all three modes of economies and is the ideal candidate for all future studies involving poverty measure. Further, zones B and C appear to suggest an inverted-U hypothesis similar to Kuznets [23] and

[3] A hyper-geometric series is an algebraic power series in which the ratio of successive coefficients r_n/r_{n-1} is a rational function of n. The hyper-geometric series that we are using here is due to Gauss and has the mathematical definition ${}_2F_1(a,b;c;z) = (\Gamma(c)/\Gamma(b)\Gamma(c-b))\int_0^1 \mathrm{d}t\, t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}$. In our case, $I(a) = (b^{-1-a}/(1+a))\,{}_2F_1(1, 1+a; 2+a; -K/b)$ for $b>0, b+K \geqslant 0$, $\mathrm{Re}[a] > -1$, $\mathrm{Im}[K] \neq 0$.

[4] To evaluate the inflexion points, we used the software mathematica and later checked the result using another software called maple. The results were once again cross-checked using a self-generated fortran code. All numerical results that we cite in this article have been cross-checked using three different and independent numerical techniques.

[5] The inflexion point is defined through the numerical solution of the coupled equations $\partial^2 P/\partial\mu^2 = 0$ and $\partial^2 P/\partial s^2 = 0$, where $s = \sigma^2$. Out of the two pairs of solution, only one turns out to be physical. The other solution gives a negative value of s. We work with the physical solution only.

Gallegati [24] that poverty increases in the early stages of development and subsequently it declines with higher level of economic progress even though such development is associated with higher inequality.

4. Comparing the log-normal and pareto distributions from an axiomatic point of view

A point of fundamental significance in the definition of a poverty index is its compatibility with the axiomatic framework as propounded by Sen [2] and Kakwani [6]. A detailed discussion of the topic can be had in the article by Kumar et al. [8]. In the following, our attempt would be to show that the poverty index defined by the pareto distribution satisfies all three axioms—the *monotonicity axiom*, the *transfer axiom* and the *transfer sensitivity axiom*—whereas the log-normal distribution falls sort of this requirement. Our contention is that this axiomatic analysis establishes without doubts the fundamental reason for the qualitative as well as quantitative supremacy of the pareto poverty distribution over the log-normal one, in that one is mathematically more completely defined than the other one. We start our analysis by quoting the three axioms in the line of Kumar et al. [8] and then providing a graphical appraisal of these with respect to the two distributions under consideration.

Monotonicity axiom: All other parameters remaining unchanged, an increase in the mean income of a family below the poverty line contributes to decrease the poverty.

Fig. 5 clearly shows that while the pareto poverty index satisfies the monotonicity axiom, the log-normal one does not.

Transfer axiom: All other parameters remaining unchanged, transfer of income from a family below the poverty line to one above contributes in increasing the poverty.

Fig. 6 above shows that both the pareto and the log-normal poverty indices satisfy the transfer axiom.

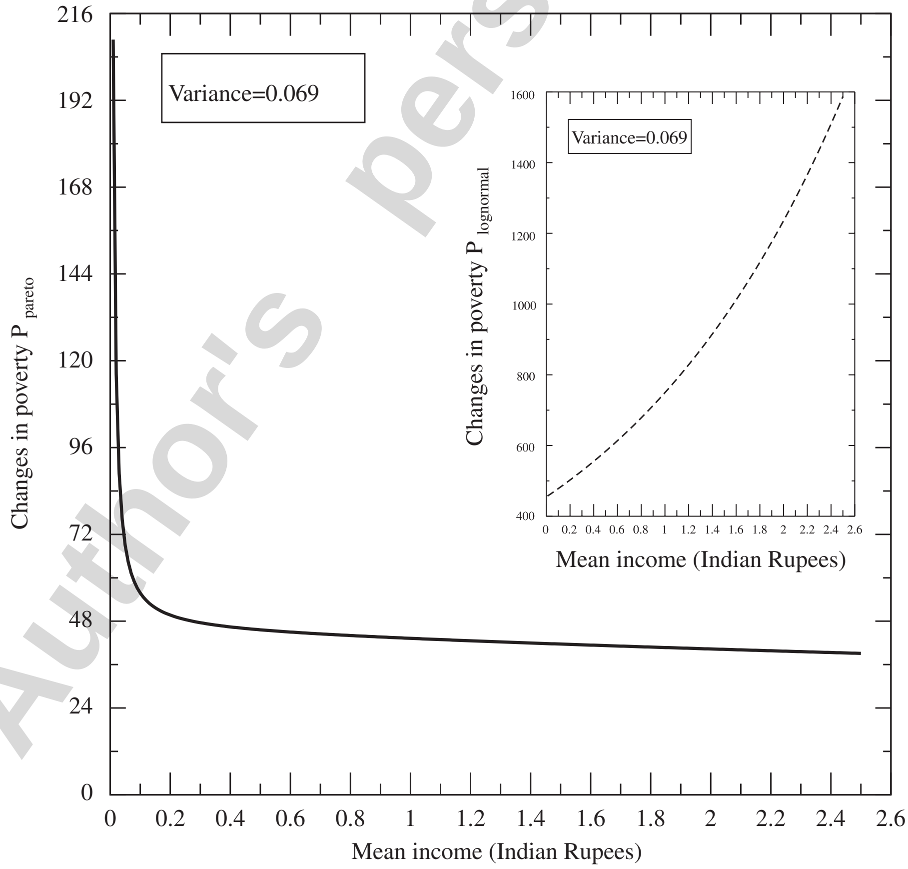

Fig. 5. Changes in the poverty function for the pareto income distribution with the variation in mean income using a fixed value for the variance (= 0.069, the inflexion point value). The inset shows an identical variation but with the log-normal distribution used instead of the pareto.

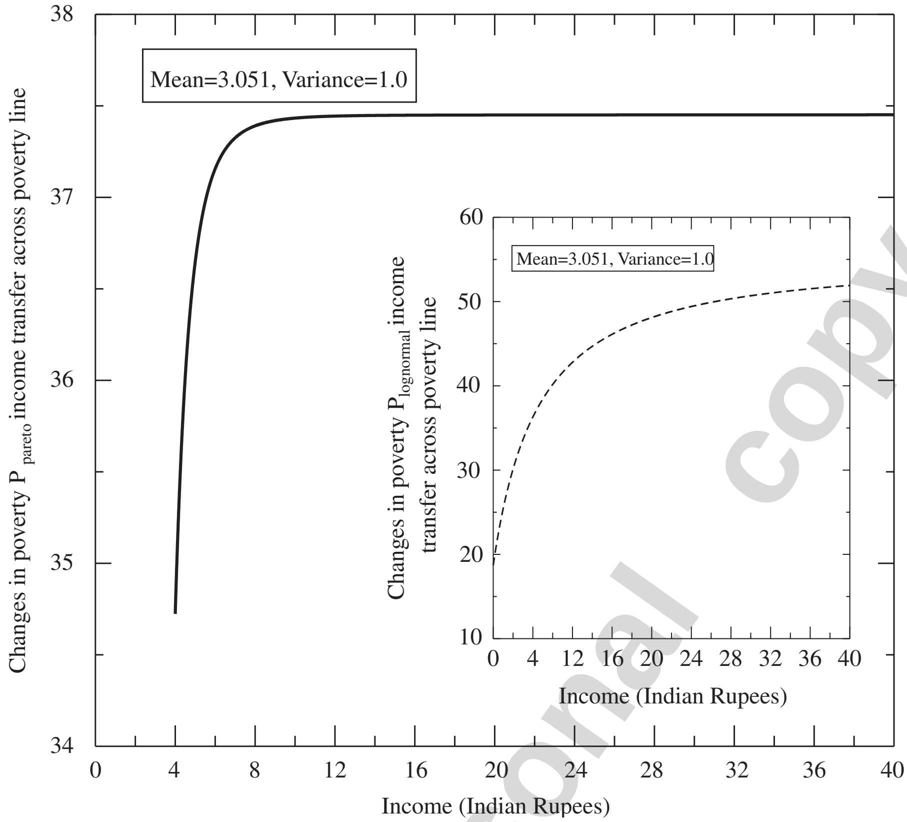

Fig. 6. Changes in the poverty function for the pareto income distribution due to a transfer of wealth across the poverty line for fixed mean (= 3.051, a value close to the inflexion point) and variance (= 1.0, a value greater than at the inflexion point). The inflexion point defines the line of poverty. The inset shows an identical variation for the log-normal distribution for the same set of parameters.

Transfer sensitivity axiom: All other parameters remaining unchanged, an exchange of mean income between two families below the poverty line contribute to decreasing the magnitude of poverty increase.

Fig. 7 above again shows that while the pareto poverty index satisfies the monotonicity axiom, the log-normal one does not.

We end this section by pointing to another important property of the pareto poverty index. As observed by Atkinson [4] and others [8], the definition of a poverty line is somewhat subjective and need not have a proper quantitative basis. In fact, it has been suggested in the same references that this parameter can be chosen based on administrative criteria. This naturally renders an unsolicited coarse-graining in an otherwise solid mathematical foundation and it is our claim that the usage of a pareto distribution for the income distribution allows us to get around this problem in a convincing way based on the mathematical structure of the distribution itself. In other words, one need not have an external parameter to define the poverty line, the pareto distribution already has an inflexion point (in fact two but the second one is non-physical as already explained before) inbuilt that we propose as a definition of the poverty line. The advantage lies with the parameters that it depends on, namely the mean and the variance of the distribution, which are already defined for the system. The Figs. 5–7 have been defined with respect to such a poverty line although, of course, the results are general and hold true for any other suitably defined poverty line as well. We should like to make it a point though that our choice, indeed suggestion, for the poverty line is not an unique one but its strength lies in its generality which again depends only on the distribution itself.

5. Conclusion

This paper made use of a poverty function, which is different from the conventional poverty indices in the following manner: (1) the CD index does not depend on an arbitrarily chosen poverty line, (2) it depends on the observed and measurable consumption behavior of people, (3) the index satisfies the standard axioms of a

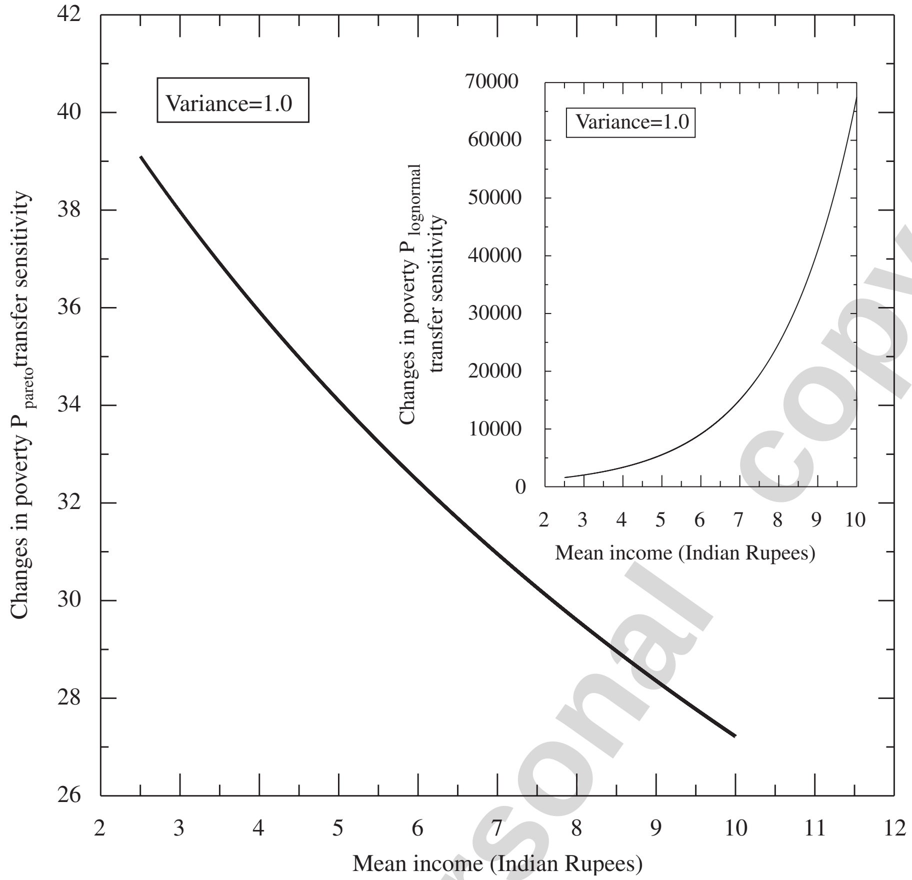

Fig. 7. Changes in the poverty function for the pareto income distribution due to a transfer of mean income below the poverty line. We use a variance (= 1.0) that is far greater than the value at the inflexion point. The inset shows an identical variation but with the log-normal distribution used instead of the pareto.

poverty index. Having used such a CD function as a measure of poverty, this paper has shown analytically that for a log-normal income distribution, an increase in mean income, ceteris paribus, will decrease poverty while an increase in the variance of the income distribution, ceteris paribus, will increase poverty although somewhat contradictory information was obtained for the limiting case of earners with extremely low variance in their income distribution. In this case, poverty was found to decrease with increasing variance for a fixed mean, while when plotted against the mean (Fig. 3), it was found to initially increase and then saturate after a critical value of the mean which we could determine theoretically.

These observations were later contrasted with observations made from a pareto distribution. Here we found that for very low earning groups in a developing economy, poverty initially decreases with increasing variance but beyond a critical value of the variance, it starts increasing later to decrease again. In the process, this defines all three economies characterized by individual parametric regimes. The conclusion that we derive from these joint analyzes is that the variance dependence of poverty is not unequivocally simplistic, in that one distribution (log-normal) predicts an increase in poverty with increasing variance (although the limiting $\sigma^2 \to 0$ case was somewhat qualitatively identical to zone B for the pareto distribution) while the other (pareto) shows the existence of an inflexion point in the poverty function. This means that the poverty-variance graph in a pareto distribution has a critical point, on one side (zone A) of which poverty decreases with increasing variance, while on the other side it is just the reverse.

A treatment of the two distributions from an axiomatic point of view, clearly shows that the pareto poverty index satisfies all three axioms [2] whereas the log-normal index falls short of this requirement. We argue that this inherent deficiency on the part of the log-normal distribution shows itself more quantitatively in the other results that we have already narrated. In the process, we propose a mathematical foundation for the poverty line, again with respect to our preferred choice, the pareto distribution. This allows one the luxury of being able to define a poverty line without resorting to any external arbitrary parameter, this special poverty line being already defined through the distribution parameters itself. There might be other choices for a poverty line, but seldom a quantitative one.

Our net contribution in this article has been to prove that a pareto distribution offers the more realistic measure of poverty in a developing economy. This is because it condones the very realistic fact that for very low income groups a slight increase in the variance only serves to decrease poverty whereas for high earning groups, greater the variation in earning greater is the probability of an escalation in poverty up to another critical point, beyond which poverty declines with any further increase in variance of wealth distribution in a society. This phase seems to reflect the case of a very developed economy, one which we identify as the supra-economic behavior. In macroeconomic sense, this phase suggests that close to an equilibrium dynamics, higher inequality could contribute to higher savings and thereby higher growth and reduced poverty. As we find, such results are not altogether extraneous with a view toward the axiomatic foundation of the distribution, pareto—or the lack of it—log-normal. As a sequel to the present work, we are presently studying the non-stationary case where both income and consumption are functions of time. The objective is to have a theory that can make quantitative predictions on the poverty of a developing economy as a function of income and time.

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